Math 111: Picture Proof of the Sum Trigonometric Identities

Three particular identities are very important to the study of trigonometry. They are typically know as the sum trigonometric identities. This paper present a rectangular geometric proof of the validity of the first two of these identities, along with an algebraic proof of the last one (3).

Theorem 1. *Suppose that α and β are any two angles. Further suppose that $\tan\alpha$ and $\tan\beta$, are defined for α and β. It follows that*

Must Memorize!

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha, \tag{1}$$

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta, \ \textit{and} \tag{2}$$

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}. \tag{3}$$

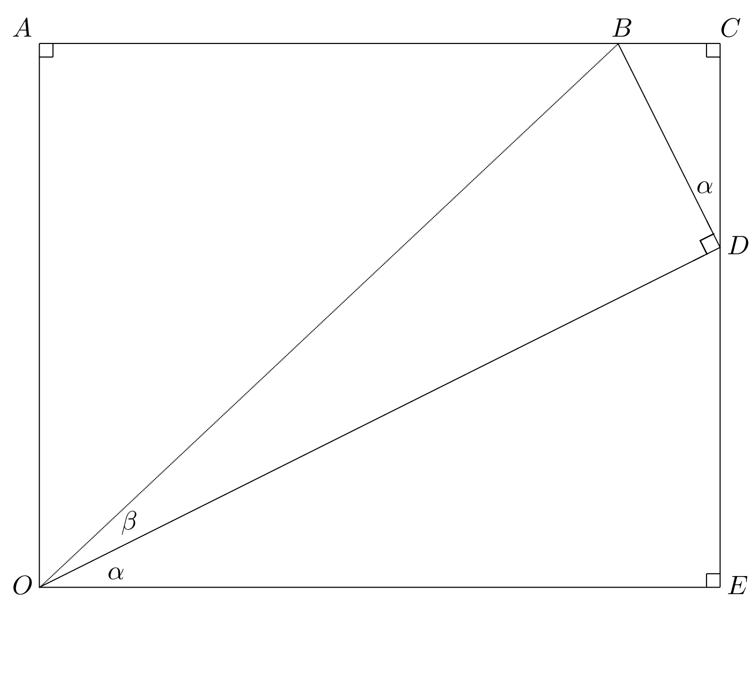
Proof. Draw two right triangles on top of each other in the following manner:

- α is the angle of the first triangle, and β is the angle of the second triangle.
- Label the triangles as ΔOED and ΔODB.
- The hypotenuse of ΔOED is the adjacent leg of ΔODB.
- Next, add points A and C to create rectangle $OECA$ and right triangles ΔOAB and ΔDCB as shown below.
- Assume that the length of OB is one ($\overline{OB} = 1$).

Then from the four triangles, the following statements fall in succession from the definition of sin and cos:

1. From ΔODB, $\overline{OD} = \cos\beta$ and $\overline{BD} = \sin\beta$ (remember that $OB = 1$).
2. From ΔDCB, $\overline{BC} = \sin\alpha\sin\beta$ and $\overline{CD} = \sin\beta\cos\alpha$
3. From ΔOED, $\overline{DE} = \sin\alpha\cos\beta$ and $\overline{OE} = \cos\alpha\cos\beta$
4. Since AC is parallel to OE, then $\angle OBA = \alpha + \beta$
5. From ΔOAB, $\overline{AB} = \cos(\alpha+\beta)$, and $\overline{OA} = \sin(\alpha+\beta)$.
6. It follows from $\overline{OA} = \overline{EC}$, $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$.
7. Finally, because $\overline{AC} = \overline{OE}$, it follows that $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$.

Turn the page for the tangent proof. A labeled figure like below with all the parts labeled is on the next page.



Finally, to prove (3), divide (1) by (2), and then divide both the top and bottom of the result by $\cos\alpha\cos\beta$ as follows:

$$\tan(\alpha+\beta) = \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)} = \frac{\sin\alpha\cos\beta + \sin\beta\cos\alpha}{\cos\alpha\cos\beta - \sin\beta\sin\alpha}$$

$$= \frac{\dfrac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta} + \dfrac{\sin\beta\cos\alpha}{\cos\alpha\cos\beta}}{\dfrac{\cos\alpha\cos\beta}{\cos\alpha\cos\beta} - \dfrac{\sin\beta\sin\alpha}{\cos\alpha\cos\beta}}$$

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

$\square$