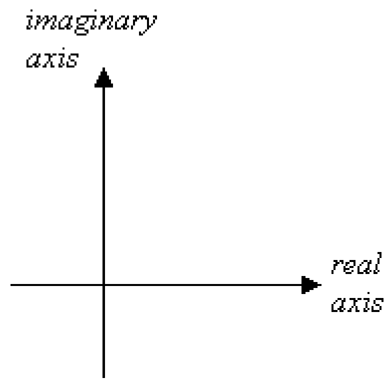
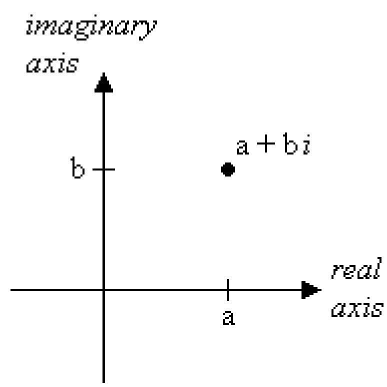
Complex Numbers in Polar Form; DeMoivre's Theorem

So far you have plotted points in both the rectangular and polar coordinate plane. We will now examine the complex plane which is used to plot complex numbers through the use of a real axis (horizontal) and an imaginary axis (vertical).



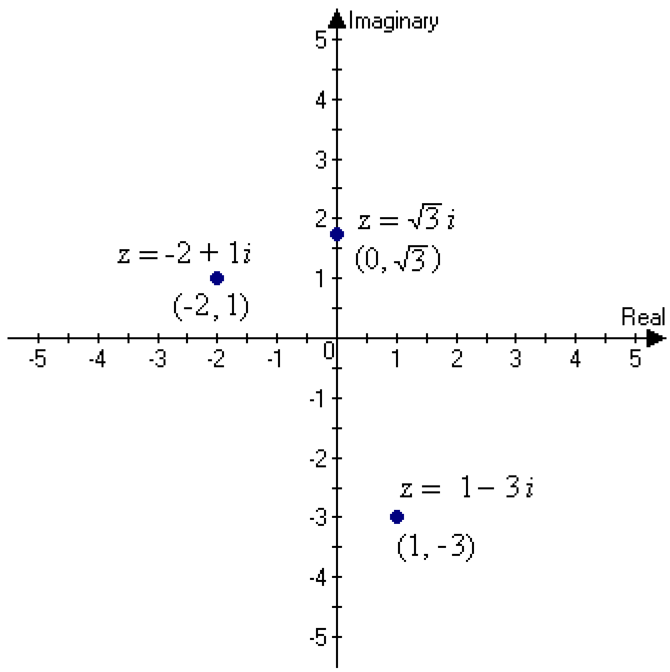
A point (a,b) in the complex plane would be represented by the complex number z = a + bi.



Example 1: Plot the following complex numbers in the complex plane.

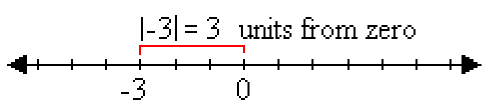
a.) $-2 + i$ b.) $1 - 3i$ c.) $\sqrt{3}i$

Solution:



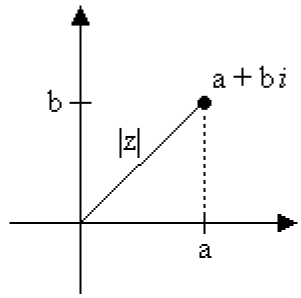
When we were dealing with real numbers, the absolute value of a real number represented the distance of the number from zero on the number line.

$|-3| = 3$



The same is true of the absolute value of a complex number. However, now the point is not simply on the real number line. There is a horizontal and vertical component for the complex number.

If we were to draw a line from the origin to the complex number z in the complex plane, we can see that its distance from the origin (absolute value) would be the hypotenuse of a right triangle and can be determined by using the Pythagorean Theorem.



$$c^2 = a^2 + b^2$$
$$\sqrt{c^2} = \sqrt{a^2 + b^2}$$
$$|c| = \sqrt{a^2 + b^2}$$
$$|z| = \sqrt{a^2 + b^2}$$

Therefore, the absolute value of complex number is $|z| = |a + bi| = \sqrt{a^2 + b^2}$.

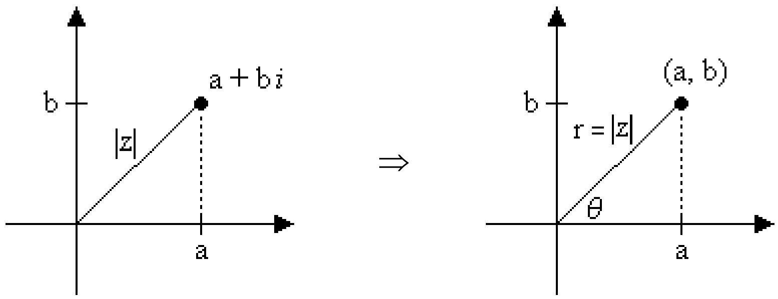
Example 2: Determine the absolute value of the complex number $-2\sqrt{3} - 2i$.

Solution:

$$|z| = \sqrt{a^2 + b^2}$$
$$|z| = \sqrt{\left(-2\sqrt{3}\right)^2 + \left(-2\right)^2}$$
$$|z| = \sqrt{12 + 4}$$
$$|z| = \sqrt{16}$$
$$|z| = 4$$

$$\left|-2\sqrt{3} - 2i\right| = 4$$

A complex number in the form of a + bi, whose point is (a, b), is in rectangular form and can therefore be converted into polar form just as we need with the points (x, y). The relationship between a complex number in rectangular form and polar form can be made by letting θ be the angle (in standard position) whose terminal side passes through the point (a, b).



$$\sin\theta = \frac{b}{r} \qquad \cos\theta = \frac{a}{r} \qquad \tan\theta = \frac{b}{a}$$

$$r\ \sin\theta = b \qquad r\ \cos\theta = a$$

$$r = |\,z\,| = \sqrt{a^2 + b^2}$$

Using these relationships, we can convert the complex number z from its rectangular form to its polar form.

$z = a + bi$
$z = (r \cos\theta) + (r \sin\theta)i$
$z = r \cos\theta + r\, i \sin\theta$
$z = r (\cos\theta + i \sin\theta)$

Example 3: Plot the complex number $z = -\sqrt{3} + i$ in the complex plane and then write it in its polar form.

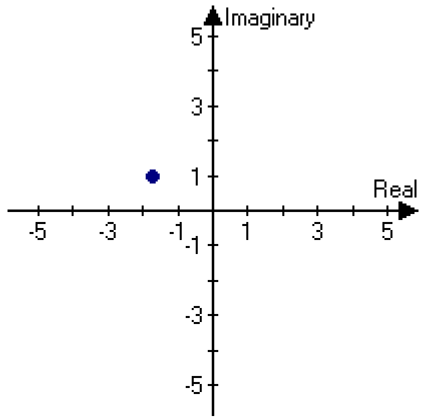
Solution:

Find r

$$r = \sqrt{a^2 + b^2}$$
$$r = \sqrt{\left(-\sqrt{3}\right)^2 + (1)^2}$$
$$r = \sqrt{3 + 1}$$
$$r = \sqrt{4}$$
$$r = 2$$

Example 3 (Continued):

Plot the complex number to determine the quadrant in which it lies



The angle θ would be in quadrant II

Find θ

$$\tan\theta = \frac{b}{a}$$
$$\tan\theta = \frac{1}{-\sqrt{3}}$$
$$\tan\theta = -\frac{\sqrt{3}}{3}$$

$\tan\frac{\pi}{6} = \frac{\sqrt{3}}{3}$ so the reference angle of $\frac{\pi}{6}$ would be subtracted from π to get the value of θ.

$$\theta = \pi - \frac{\pi}{6}$$
$$\theta = \frac{5\pi}{6}$$

Write the complex number in its polar form

z = r (cos θ + *i* sin θ)

$z = 2\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$

Example 4: Write the complex number $z = 5(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})$ in its rectangular form and then plot it in the complex plane.

Solution:

Evaluate cos and sin at the value of theta

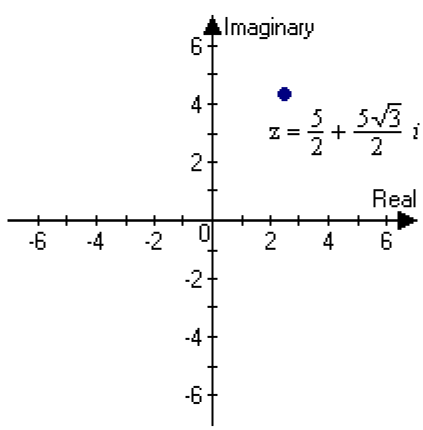
$$\cos\frac{\pi}{3} = \frac{1}{2}$$

$$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

Substitute in the exact values of cos and sin to find the rectangular form

$$z = 5(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})$$
$$z = 5(\frac{1}{2} + i\frac{\sqrt{3}}{2})$$
$$z = \frac{5}{2} + \frac{5\sqrt{3}}{2}\,i$$

Plot the complex number



Mathematical operations on complex numbers in polar form

If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ then their product z_1z_2 is given by:

$$z_1z_2 = r_1r_2\left[\cos\left(\theta_1+\theta_2\right)+i\sin\left(\theta_1+\theta_2\right)\right]$$

If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ then their quotient $\frac{z_1}{z_2}$ is given by:

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\left[\cos\left(\theta_1-\theta_2\right)+i\sin\left(\theta_1-\theta_2\right)\right]$$

If $z = r(\cos\theta + i\sin\theta)$ then raising the complex number to a power is given by DeMoivre's Theorem:

$$z^n = r^n\left(\cos n\theta + i\sin n\theta\right)$$; where n is a positive integer

If $w = r(\cos\theta + i\sin\theta)$ where w ≠ 0 then w has n distinct complex nth roots given by DeMoivre's Theorem:

$$z_k = \sqrt[n]{r}\left[\cos\left(\frac{\theta+2\pi k}{n}\right)+i\sin\left(\frac{\theta+2\pi k}{n}\right)\right]$$ (radian measurement)

or

$$z_k = \sqrt[n]{r}\left[\cos\left(\frac{\theta+360°k}{n}\right)+i\sin\left(\frac{\theta+360°k}{n}\right)\right]$$ (degree measurement)

where k = 0, 1, 2, …, n – 1

Example 5: Find the product of the complex numbers $z_1 = 4(\cos 32° + i\sin 32°)$ and $z_2 = 3(\cos 61° + i\sin 61°)$. Leave the answer in polar form.

Solution:

$$z_1z_2 = r_1r_2\left[\cos\left(\theta_1+\theta_2\right)+i\sin\left(\theta_1+\theta_2\right)\right]$$
$$z_1z_2 = (4)(3)\left[\cos\left(32°+61°\right)+i\sin\left(32°+61°\right)\right]$$
$$z_1z_2 = 12\left[\cos 93° + i\sin 93°\right]$$

Example 6: Find the quotient of the complex numbers $z_1 = 12(\cos 84° + i \sin 84°)$ and $z_2 = 3(\cos 35° + i \sin 35°)$. Leave the answer in polar form.

Solution:

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\left[\cos\left(\theta_1 - \theta_2\right) + i \sin\left(\theta_1 - \theta_2\right)\right]$$
$$\frac{z_1}{z_2} = \frac{12}{3}\left[\cos\left(84° - 35°\right) + i \sin\left(84° - 35°\right)\right]$$
$$\frac{z_1}{z_2} = 4\left[\cos 49° + i \sin 49°\right]$$

Example 7: Use DeMoivre's Theorem to find the 5th power of the complex number $z = 2(\cos 24° + i \sin 24°)$. Express the answer in the rectangular form a + bi.

Solution:

$$z^n = r^n\left(\cos n\theta + i \sin n\theta\right)$$
$$z^5 = 2^5\left[\cos 5(24°) + i \sin 5(24°)\right]$$
$$z^5 = 32\left(\cos 120° + i \sin 120°\right)$$
$$z^5 = 32\left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right)$$
$$z^5 = 16\sqrt{3} + 16i$$

Example 8: Use DeMoivre's Theorem to find the 3rd power of the complex number $z = (2 + 2i)$. Express the answer in the rectangular form a + bi.

Solution:

Since the complex number is in rectangular form we must first convert it into polar form before using DeMoivre's Theorem.

Find r

$$r = \sqrt{a^2 + b^2}$$
$$r = \sqrt{2^2 + 2^2}$$
$$r = \sqrt{4 + 4}$$
$$r = \sqrt{8}$$
$$r = 2\sqrt{2}$$

Example 8 (Continued):

Find θ

$$\tan\theta = \frac{b}{a}$$
$$\tan\theta = \frac{2}{2}$$
$$\tan\theta = 1$$
$$\theta = \frac{\pi}{4}$$

Apply DeMoivre's Theorem

$$z^n = r^n\left(\cos n\theta + i\sin n\theta\right)$$
$$z^3 = \left(2\sqrt{2}\right)^3\left[\cos 3\left(\frac{\pi}{4}\right) + i\sin 3\left(\frac{\pi}{4}\right)\right]$$
$$z^3 = 16\sqrt{2}\left[\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right]$$
$$z^3 = 16\sqrt{2}\left(-\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)$$
$$z^3 = -16 + 16i$$

Example 9: Find the complex cube roots of 8(cos 60° + *i* sin 60°).

Solution:

Let k = 0 and n = 3 to find the first complex cube root

$$z_k = \sqrt[n]{r}\left[\cos\left(\frac{\theta + 360°k}{n}\right) + i\sin\left(\frac{\theta + 360°k}{n}\right)\right]$$
$$z_0 = \sqrt[3]{8}\left[\cos\left(\frac{60° + 360°(0)}{3}\right) + i\sin\left(\frac{60° + 360°(0)}{3}\right)\right]$$
$$z_0 = 2\left[\cos\left(\frac{60°}{3}\right) + i\sin\left(\frac{60°}{3}\right)\right]$$
$$z_0 = 2\left(\cos 20° + i\sin 20°\right)$$

Example 9 (Continued):

Let k = 1 and n = 3 to find the second complex cube root

$$z_k = \sqrt[n]{r}\left[\cos\left(\frac{\theta + 360°k}{n}\right) + i\sin\left(\frac{\theta + 360°k}{n}\right)\right]$$

$$z_1 = \sqrt[3]{8}\left[\cos\left(\frac{60° + 360°(1)}{3}\right) + i\sin\left(\frac{60° + 360°(1)}{3}\right)\right]$$

$$z_1 = 2\left[\cos\left(\frac{420°}{3}\right) + i\sin\left(\frac{420°}{3}\right)\right]$$

$$z_1 = 2\left(\cos 140° + i\sin 140°\right)$$

Let k = 2 and n = 3 to find the third complex cube root

$$z_k = \sqrt[n]{r}\left[\cos\left(\frac{\theta + 360°k}{n}\right) + i\sin\left(\frac{\theta + 360°k}{n}\right)\right]$$

$$z_2 = \sqrt[3]{8}\left[\cos\left(\frac{60° + 360°(2)}{3}\right) + i\sin\left(\frac{60° + 360°(2)}{3}\right)\right]$$

$$z_2 = 2\left[\cos\left(\frac{780°}{3}\right) + i\sin\left(\frac{780°}{3}\right)\right]$$

$$z_2 = 2\left(\cos 260° + i\sin 260°\right)$$