Complex Numbers

$$i = \sqrt{-1} \qquad i^2 = -1 \qquad i^3 = -i \qquad i^4 = 1$$

$\sqrt{-a} = i\sqrt{a}, \quad a \geq 0$

$(a+bi)(a-bi) = a^2 + b^2$

$(a+bi) + (c+di) = a + c + (b+d)i$

$|a+bi| = \sqrt{a^2+b^2}$ **Complex Modulus**

$(a+bi) - (c+di) = a - c + (b-d)i$

$\overline{(a+bi)} = a - bi$ **Complex Conjugate**

$(a+bi)(c+di) = ac - bd + (ad+bc)i$

$\overline{(a+bi)}(a+bi) = |a+bi|^2$

DeMoivre's Theorem

Let $z = r(\cos\theta + i\sin\theta)$, and let n be a positive integer.
Then:

$$z^n = r^n(\cos n\theta + i \sin n\theta).$$

Example: Let $z = 1 - i$, find z^6.

Solution: First write z in polar form.

$r = \sqrt{(1)^2 + (-1)^2} = \sqrt{2}$

$\theta = arg(z) = \tan^{-1}\left(\frac{-1}{1}\right) = -\frac{\pi}{4}$

Polar Form: $z = \sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)$

Applying DeMoivre's Theorem gives :

$$\begin{aligned} z^6 &= \left(\sqrt{2}\right)^6\left(\cos\left(6\cdot -\frac{\pi}{4}\right) + i\sin\left(6\cdot -\frac{\pi}{4}\right)\right) \\ &= 2^3\left(\cos\left(-\frac{3\pi}{2}\right) + i\sin\left(-\frac{3\pi}{2}\right)\right) \\ &= 8(0 + i(1)) \\ &= 8i \end{aligned}$$

Finding the nth roots of a number using DeMoivre's Theorem

Example: Find all the complex fourth roots of 4. That is, find all the complex solutions of $x^4 = 4$.

We are asked to find all complex fourth roots of 4.
These are all the solutions (including the complex values) of the equation $x^4 = 4$.

For any positive integer n , a nonzero complex number z has exactly n distinct nth roots.
More specifically, if z is written in the trigonometric form $r(\cos\theta + i\sin\theta)$, the nth roots of z are given by the following formula.

$$(*)\ r^{\frac{1}{n}}\left(\cos\left(\frac{\theta}{n}+\frac{360°k}{n}\right)+i\sin\left(\frac{\theta}{n}+\frac{360°k}{n}\right)\right),\quad for\ \ k=0,1,2,...,n-1.$$

Remember from the previous example we need to write 4 in trigonometric form by using:

$$r=\sqrt{(a)^2+(b)^2}\qquad\text{and}\qquad\theta=arg(z)=\tan^{-1}\left(\frac{b}{a}\right).$$

So we have the complex number $a + ib = 4 + i0$.

Therefore $a = 4$ and $b = 0$

So $r=\sqrt{(4)^2+(0)^2}=4$ and
$\theta=arg(z)=\tan^{-1}\left(\frac{0}{4}\right)=0$
Finally our trigonometric form is $4 = 4(\cos 0° + i\sin 0°)$

Using the formula $(*)$ above with $n = 4$, we can find the fourth roots of $4(\cos 0° + i\sin 0°)$

- For $k=0$, $4^{\frac{1}{4}}\left(\cos\left(\frac{0°}{4}+\frac{360°*0}{4}\right)+i\sin\left(\frac{0°}{4}+\frac{360°*0}{4}\right)\right)=\sqrt{2}\left(\cos(0°)+i\sin(0°)\right)=\sqrt{2}$
- For $k=1$, $4^{\frac{1}{4}}\left(\cos\left(\frac{0°}{4}+\frac{360°*1}{4}\right)+i\sin\left(\frac{0°}{4}+\frac{360°*1}{4}\right)\right)=\sqrt{2}\left(\cos(90°)+i\sin(90°)\right)=\sqrt{2}i$
- For $k=2$, $4^{\frac{1}{4}}\left(\cos\left(\frac{0°}{4}+\frac{360°*2}{4}\right)+i\sin\left(\frac{0°}{4}+\frac{360°*2}{4}\right)\right)=\sqrt{2}\left(\cos(180°)+i\sin(180°)\right)=-\sqrt{2}$
- For $k=3$, $4^{\frac{1}{4}}\left(\cos\left(\frac{0°}{4}+\frac{360°*3}{4}\right)+i\sin\left(\frac{0°}{4}+\frac{360°*3}{4}\right)\right)=\sqrt{2}\left(\cos(270°)+i\sin(270°)\right)=-\sqrt{2}i$

Thus all of the complex roots of $x^4 = 4$ are:

$\mathbf{\sqrt{2}, \sqrt{2}i, -\sqrt{2}, -\sqrt{2}i}$.