Math 111: Derive exact value of the cosine of $2\pi/5$

First, note that if $\alpha = \frac{2\pi}{5}$, then it follows that $\cos(2\alpha) = \cos(3\alpha)$ since

$$2\alpha = 2\left(\frac{2\pi}{5}\right) = \frac{4\pi}{5} = \pi - \frac{\pi}{5}$$

$$3\alpha = 3\left(\frac{2\pi}{5}\right) = \frac{6\pi}{5} = \pi + \frac{\pi}{5}$$

and

$$\cos(2\alpha) = \cos\left(\pi - \frac{\pi}{5}\right) = \cos\pi\cos\left(\frac{\pi}{5}\right) + \sin\pi\sin\left(\frac{\pi}{5}\right) = -\cos\left(\frac{\pi}{5}\right)$$

$$\cos(3\alpha) = \cos\left(\pi + \frac{\pi}{5}\right) = \cos\pi\cos\left(\frac{\pi}{5}\right) - \sin\pi\sin\left(\frac{\pi}{5}\right) = -\cos\left(\frac{\pi}{5}\right)$$

It follows that

$$\begin{aligned} 0 &= \cos(3\alpha) - \cos(2\alpha) \\ &= 4\cos^3\alpha - 3\cos\alpha - 2\cos^2\alpha + 1 \\ &= (\cos\alpha - 1)(4\cos^2\alpha + 2\cos\alpha - 1) \end{aligned}$$

Since $\cos\alpha \neq 1$, then

$$4\cos^2 x + 2\cos x - 1 = 0.$$

Thus,

$$\cos\alpha = \frac{-2 \pm \sqrt{4 - 4(4)(-1)}}{8} = \frac{-2 \pm \sqrt{20}}{8} = \frac{-1 \pm \sqrt{5}}{4}$$

Since $0 < \frac{2\pi}{5} < \frac{\pi}{2}$, then $\cos\alpha$ is positive. It follows that

$$\cos\left(\frac{2\pi}{5}\right) = \frac{-1 + \sqrt{5}}{4}$$

It also follows that

$$\cos(2\alpha) = \cos\left(\frac{4\pi}{5}\right) = 2\cos^2\left(\frac{2\pi}{5}\right) - 1 = 2\left(\frac{-1 + \sqrt{5}}{4}\right)^2 - 1 = \frac{1}{8}\left(6 - 2\sqrt{5}\right) - 1 = \frac{3}{4} - 1 - \frac{1}{4}\sqrt{5}$$

$$\cos\left(\frac{4\pi}{5}\right) = \frac{-1 - \sqrt{5}}{4}$$

Thus,

$$\cos(3\alpha) = \cos\left(\frac{6\pi}{5}\right) = \cos(2\alpha) = \frac{-1 - \sqrt{5}}{4} \text{ and}$$

$$\cos\left(\frac{\pi}{5}\right) = -\cos\left(\frac{4\pi}{5}\right) = \frac{1 + \sqrt{5}}{4}$$