# **Math 111**: Derivation of Trigonometric Identities

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Many of the trigonometric identities can be derived in succession from the identities:

$$\sin(-\theta) = -\sin\theta, \tag{1}$$
$$\cos(-\theta) = \cos\theta, \tag{2}$$
$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha, \text{ and} \tag{3}$$
$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta. \tag{4}$$

The first and second identities indicate that sin and cos are odd and even functions, respectively. The second two are known as the sum identies.
Suppose that $\beta = -w$, then (3) simplifies to

$$\begin{aligned}
\sin(\alpha + (-w)) &= \sin\alpha\cos(-w) + \sin(-w)\cos\alpha && \text{by (3)} \\
&= \sin\alpha\cos w - \sin w\cos\alpha && \text{by (1) and (2)}
\end{aligned}$$

Since $w$ is an arbitrary label, then $\beta$ will do as well. Hence, the sine difference formula is

$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \sin\beta\cos\alpha \tag{5}$$

Similarly, equation (4) simplifies to the cosine difference formula:

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta \tag{6}$$

To find the sum identity for $\tan(\alpha+\beta)$, divide (3) by (4) as follows:

$$\tan(\alpha+\beta) = \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)} = \frac{\sin\alpha\cos\beta + \sin\beta\cos\alpha}{\cos\alpha\cos\beta - \sin\alpha\sin\beta} \tag{7}$$

Divide both the top and bottom of (7) by $\cos\alpha\cos\beta$ results with the simplified formula

$$\tan(\alpha+\beta) = \frac{\dfrac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta} + \dfrac{\sin\beta\cos\alpha}{\cos\alpha\cos\beta}}{\dfrac{\cos\alpha\cos\beta}{\cos\alpha\cos\beta} - \dfrac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}} = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} \tag{8}$$

Because $\tan(-\theta) = \frac{\sin(-\theta)}{\cos(-\theta)} = \frac{-\sin\theta}{\cos\theta} = -\tan\theta$, then it follows that

$$\tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}. \tag{9}$$

The Double Angle identities can be derived from equations (3) and (4). Suppose $\alpha = \beta = \theta$, then (3) simplifies as

$$\sin(\theta+\theta) = \sin\theta\cos\theta + \sin\theta\cos\theta.$$

Hence,

$$\sin(2\theta) = 2\sin\theta\cos\theta. \tag{10}$$

Similarly,

$$\cos 2\theta = \cos^2\theta - \sin^2\theta, \text{ and} \tag{11}$$
$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}. \tag{12}$$

The first of the Pythagorean identities can be found by setting $\alpha = \beta = \theta$ in (6). Hence,

$$\cos(\theta-\theta) = \sin\theta\sin\theta + \cos\theta\cos\theta.$$

So,

$$\sin^2\theta + \cos^2\theta = 1. \tag{13}$$

Dividing both sides of (13) by $\cos^2\theta$ yields

$$\tan^2\theta + 1 = \sec^2\theta. \tag{14}$$

Dividing both sides of (13) by $\sin^2\theta$ yields

$$1 + \cot^2\theta = \csc^2\theta. \tag{15}$$

Equations (11) and (13) can generate the Power Reducting identities. Using $\cos^2\theta = 1 - \sin^2\theta$, (11) can be written as

$$\cos(2\theta) = (1 - \sin^2\theta) - \sin^2\theta = 1 - 2\sin^2\theta.$$

Solving the above equation for $\sin^2\theta$ yields

$$\sin^2\theta = \frac{1 - \cos(2\theta)}{2}. \tag{16}$$

Similarly,

$$\cos^2\theta = \frac{1 + \cos(2\theta)}{2}, \text{ and} \tag{17}$$

$$\tan^2\theta = \frac{\sin^2\theta}{\cos^2\theta} = \frac{1 - \cos(2\theta)}{1 + \cos(2\theta)}. \tag{18}$$

The product identities can be found using equations (3) through (6). For example, adding (3) and (5) yields

$$\begin{aligned}\sin(\alpha - \beta) + \sin(\alpha + \beta) &= \sin\alpha\cos\beta - \sin\beta\cos\alpha + \sin\alpha\cos\beta + \sin\beta\cos\alpha \\ \sin(\alpha - \beta) + \sin(\alpha + \beta) &= 2\sin\alpha\cos\beta.\end{aligned}$$

Hence,

$$\sin\alpha\cos\beta = \frac{1}{2}\left[\sin(\alpha - \beta) + \sin(\alpha + \beta)\right]. \tag{19}$$

Similarly,

$$\cos\alpha\sin\beta = \frac{1}{2}\left[\sin(\alpha - \beta) + \sin(\alpha + \beta)\right], \tag{20}$$

$$\cos\alpha\cos\beta = \frac{1}{2}\left[\cos(\alpha - \beta) + \cos(\alpha + \beta)\right], \text{ and} \tag{21}$$

$$\sin\alpha\sin\beta = \frac{1}{2}\left[\cos(\alpha - \beta) - \cos(\alpha + \beta)\right]. \tag{22}$$

Substituting $\alpha = \frac{u+v}{2}$ and $\beta = \frac{u-v}{2}$ into (19) yields:

$$\begin{aligned} & \frac{1}{2}\left[\sin(\alpha - \beta) + \sin(\alpha + \beta)\right] = \sin\alpha\cos\beta \\ \Longrightarrow\quad & \sin(\alpha - \beta) + \sin(\alpha + \beta) = 2\sin\alpha\cos\beta \\ \Longrightarrow\quad & \sin\left(\frac{u+v}{2} - \frac{u-v}{2}\right) + \sin\left(\frac{u+v}{2} + \frac{u-v}{2}\right) = 2\sin\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right) \\ \Longrightarrow\quad & \sin u + \sin v = 2\sin\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right)\end{aligned}$$

Since $u$ and $v$ are arbitrary labels, then $\alpha$ and $\beta$ will do just as well. Hence,

$$\sin\alpha + \sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) \tag{23}$$

Similarly, replacing $\alpha$ by $\frac{\alpha+\beta}{2}$ and $\beta$ by $\frac{\alpha-\beta}{2}$ into (20), (21), and (22) yields

$$\sin\alpha - \sin\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right) \tag{24}$$

$$\cos\alpha + \cos\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) \tag{25}$$

$$\cos\alpha - \cos\beta = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right) \tag{26}$$