Linearization of Powers of Sine and Cosine

Use of DeMoivre's Theorem and the Binomial Theorem allows the development of identities which reduce powers of sine and cosine into linear functions of $\cos n\theta$ and $\sin n\theta$.

This paper will illustrate a method to reduce the powers on the trig functions. Applications include integration of powers of sine and cosine.

Definition. *A general complex number $z = a + bi$ can be written in polar form as*

$$z = r(\cos\theta + i\sin\theta)$$

Suppose $r = 1$. Then it follows by De'Moivre's Theorem that $z^n = \cos n\theta + i\sin n\theta$. In particular,

$$\begin{aligned} z^n &= \cos n\theta + i\sin n\theta \\ z^{-n} &= \cos n\theta - i\sin n\theta \end{aligned}$$

It follows that

$$z^n + z^{-n} = 2\cos n\theta \text{ and} \tag{1}$$

$$z^n - z^{-n} = 2i\sin n\theta \tag{2}$$

Theorem. *Powers of $\cos\theta$ can be written as linear combinations of $\cos k\theta$. Since $2\cos\theta = z + \frac{1}{z}$, then we can expand $(z + \frac{1}{z})^n$ using the binomial theorem:*

$$(2\cos\theta)^n = 2^n\cos^n\theta = \left(z + \frac{1}{z}\right)^n = \sum_{j=0}^{n}\binom{n}{j} z^j \left(\frac{1}{z}\right)^{n-j} = \sum_{i=0}^{n}\binom{n}{j} z^{2j-n}$$

Gathering the terms above in groups of z and $\frac{1}{z}$ to the same power. Then we apply the formula in (1).

Example 1

$$\begin{aligned} 2^8\cos^8\theta &= \left(z + \frac{1}{z}\right)^8 \\ &= z^8 + 8z^7\left(\frac{1}{z}\right) + 28z^6\left(\frac{1}{z}\right)^2 + 56z^5\left(\frac{1}{z}\right)^3 + 70z^4\left(\frac{1}{z}\right)^4 + 56z^3\left(\frac{1}{z}\right)^5 + 28z^2\left(\frac{1}{z}\right)^6 + 8z^1\left(\frac{1}{z}\right)^7 + \left(\frac{1}{z}\right)^8 \\ &= z^8 + 8z^6 + 28z^4 + 56z^2 + 70 + 56z^{-2} + 28z^{-4} + 8z^{-6} + z^{-8} \end{aligned}$$

$$\begin{aligned} &= \left(z^8 + z^{-8}\right) + 8\left(z^6 + z^{-6}\right) + 28\left(z^4 + z^{-4}\right) + 56\left(z^2 + z^{-2}\right) + 70 \\ &= \left(2\cos 8\theta\right) + 8\left(2\cos 6\theta\right) + 28\left(2\cos 4\theta\right) + 56\left(2\cos 2\theta\right) + 70 \\ &= \frac{1}{128}\cos 8\theta + \frac{8}{128}\cos 6\theta + \frac{28}{128}\cos 4\theta + \frac{56}{128}\cos 2\theta + \frac{70}{256} \end{aligned}$$

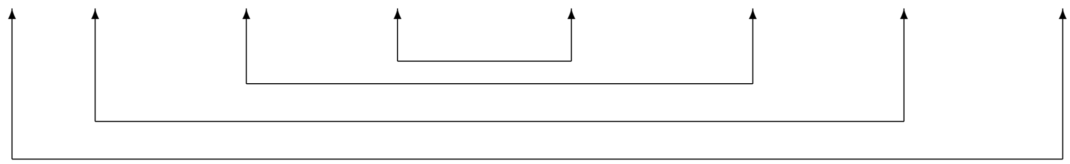
So it follows that

$$\cos^8\theta = \frac{1}{128}\cos 8\theta + \frac{1}{16}\cos 6\theta + \frac{7}{32}\cos 4\theta + \frac{7}{16}\cos 2\theta + \frac{35}{128}$$

In general, powers of cosine will simplify to a linear function of $\cos k\theta$. When the exponent is an even power, then there will be no y-intercept. Here's an example of that case. Note that the middle term is doubled on these, whereas, when n is odd, the middle term is not.

Example 2

$$
\begin{aligned}
2^7\cos^7\theta &= \left(z+\frac{1}{z}\right)^7 \\
&= z^7 \quad + 7z^6\left(\frac{1}{z}\right) + 21z^5\left(\frac{1}{z}\right)^2 \quad + 35z^4\left(\frac{1}{z}\right)^3 \quad + 35z^3\left(\frac{1}{z}\right)^4 \quad + 21z^2\left(\frac{1}{z}\right)^5 \quad + 7z^1\left(\frac{1}{z}\right)^6 \quad + \left(\frac{1}{z}\right)^7 \\
&= z^7 \quad + 7z^5 \qquad + 21z^3 \qquad + 35z \qquad + 35z^{-1} \qquad + 21z^{-3} \qquad + 7z^{-5} \qquad + z^{-7}
\end{aligned}
$$



$$
\begin{aligned}
&= \left(z^7+z^{-7}\right) + 7\left(z^5+z^{-5}\right) + 21\left(z^3+z^{-3}\right) + 35\left(z+z^{-1}\right) \\
&= \left(2\cos 7\theta\right) + 7\left(2\cos 5\theta\right) + 21\left(2\cos 3\theta\right) + 35\left(2\cos\theta\right) \\
&= \frac{1}{64}\cos 8\theta + \frac{7}{64}\cos 6\theta + \frac{21}{64}\cos 4\theta + \frac{35}{64}\cos 2\theta
\end{aligned}
$$

So it follows that

$$\cos^7\theta = \frac{1}{64}\cos 7\theta + \frac{7}{64}\cos 5\theta + \frac{21}{64}\cos 3\theta + \frac{35}{64}\cos\theta$$

Working with powers of sines is not as straight forward. When the power is odd, we will get linear combinations of $\sin k\theta$, but when the power is even, we will get linear combinations of $\cos k\theta$, which isn't initially evident. Try it out!

Even powers of Sine reduce to a linear combination of $cosk\theta$

There are four total possibilities. We have illustrated that the first and second cases (cosine to an odd or even power) leads to a result was a linear combination of $\cos k\theta$. Continuing this, we now investigate powers of sine. We will find that the result will be linear combinations of $\sin k\theta$ (when the power is odd) AND $\cos k\theta$ (when the power is even). Initially, it seems like what happened to cosine will happen for sine. However, when we have even powers of sin, it will lead to a linear combination of $\cos k\theta$. Let's illustrate that case. Let $n = 2m$, where m is an integer. It follows that

$$
\begin{aligned}
2^{2m}\cdot i^{2m}\cdot\sin^{2m}\theta &= \sum_{k=0}^{2m}\binom{2m}{k}z^k(-z^{-1})^{2m-k} = \sum_{k=0}^{2m}\underbrace{(-1)^{2m-k}}_{\text{alternates}}\binom{2m}{k}z^k z^{-2m+k} = \sum_{k=0}^{2m}(-1)^k\binom{2m}{k}z^{-2m+2k} \\
&= \sum_{k=0}^{2m}(-1)^k\binom{2m}{k}z^{2(k-m)}
\end{aligned}
$$

Reindex the sum with $w = k - m$. This gives

$$
\begin{aligned}
&= \sum_{w=-m}^{m}(-1)^{m+w}\binom{2m}{m+w}z^{2w} \\
&= \sum_{w=-m}^{-1}(-1)^{m+w}\binom{2m}{m+w}z^{2w} + \sum_{w=0}^{0}(-1)^{m+w}\binom{2m}{m+w}z^{2w} + \sum_{w=1}^{m}(-1)^{m+w}\binom{2m}{m+w}z^{2w}
\end{aligned}
$$

Bringing the middle term to the front (the one with the index of zero), and reindex the first sum with $y = -w$ yields

$$= (-1)^m\binom{2m}{m} + \sum_{y=1}^{m}(-1)^{m-y}\binom{2m}{m-y}z^{-2y} + \sum_{w=1}^{m}(-1)^{m+w}\binom{2m}{m+w}z^{2w}$$

Next, note that

$$\binom{2m}{m+w} = \frac{(2m)!}{(2m-m-w)!(m+w)!} = \frac{(2m)!}{(m-w)!(m+w)!} = \binom{2m}{m-w}$$

Note further that $(-1)^{m-w} = (-1)^{m+w}$ and the first sum can use w as its index instead of y. Thus,

$$\begin{aligned}
2^{2m} \cdot i^{2m} \cdot \sin^{2m}\theta &= (-1)^m \binom{2m}{m} + \sum_{y=1}^{m} (-1)^{m-y} \binom{2m}{m-y} z^{-2y} + \sum_{w=1}^{m} (-1)^{m+w} \binom{2m}{m+w} z^{2w} \\
&= (-1)^m \binom{2m}{m} + \sum_{w=1}^{m} (-1)^{m-w} \binom{2m}{m-w} z^{-2w} + \sum_{w=1}^{m} (-1)^{m+w} \binom{2m}{m+w} z^{2w} \\
&= (-1)^m \binom{2m}{m} + \sum_{w=1}^{m} (-1)^{m+w} \binom{2m}{m+w} z^{-2w} + \sum_{w=1}^{m} (-1)^{m+w} \binom{2m}{m+w} z^{2w}
\end{aligned}$$

Now combine the two sums into one sum and factor out $(z^{2w} + z^{-2w})$.

$$= (-1)^m \binom{2m}{m} + \sum_{w=1}^{m} (-1)^{m+w} \binom{2m}{m+w} (z^{-2w} + z^{2w})$$

Since $z^n + z^{-n} = 2\cos n\theta$, then

$$\begin{aligned}
&= (-1)^m \binom{2m}{m} + \sum_{w=1}^{m} (-1)^{m+w} \binom{2m}{m+w} (2\cos 2w\theta) \\
2^{2m} \cdot i^{2m} \cdot \sin^{2m}\theta &= (-1)^m \binom{2m}{m} + 2(-1)^m \sum_{w=1}^{m} (-1)^{w} \binom{2m}{m+w} \cos 2w\theta \\
2^{2m} \cdot (-1)^m \cdot \sin^{2m}\theta &= (-1)^m \binom{2m}{m} + 2(-1)^m \sum_{w=1}^{m} (-1)^{w} \binom{2m}{m+w} \cos 2w\theta \\
2^{2m} \cdot \sin^{2m}\theta &= \binom{2m}{m} + 2 \sum_{w=1}^{m} (-1)^{w} \binom{2m}{m+w} \cos 2w\theta
\end{aligned}$$

Finally,

$$\sin^{2m}\theta = \frac{1}{2^{2m}} \binom{2m}{m} + \frac{1}{2^{2m-1}} \sum_{w=1}^{m} (-1)^{w} \binom{2m}{m+w} \cos 2w\theta$$

Odd powers of sine lead to linear combinations of $\sin k\theta$ instead (proof omitted).

Table of powers of Sine reduced to Linear combinations

Combining all of them into one table, here are the results for powers of 10 or less:

$$\begin{aligned}
\sin^2\theta &= -\frac{1}{2}\cos 2\theta + \frac{1}{2} \\
\sin^3\theta &= -\frac{1}{4}\sin 3\theta + \frac{3}{4}\sin\theta \\
\sin^4\theta &= \frac{1}{8}\cos 4\theta - \frac{1}{2}\cos 2\theta + \frac{3}{8} \\
\sin^5\theta &= \frac{1}{16}\sin 5\theta - \frac{5}{16}\sin 3\theta + \frac{5}{8}\sin\theta \\
\sin^6\theta &= -\frac{1}{32}\cos 6\theta + \frac{3}{16}\cos 4\theta - \frac{15}{32}\cos 2\theta + \frac{5}{16} \\
\sin^7\theta &= -\frac{1}{64}\sin 7\theta + \frac{7}{64}\sin 5\theta - \frac{21}{64}\sin 3\theta + \frac{35}{64}\sin\theta \\
\sin^8\theta &= \frac{1}{128}\cos 8\theta - \frac{1}{16}\cos 6\theta + \frac{7}{32}\cos 4\theta - \frac{7}{16}\cos 2\theta + \frac{35}{128} \\
\sin^9\theta &= \frac{1}{256}\sin 9\theta - \frac{9}{256}\sin 7\theta + \frac{9}{64}\sin 5\theta - \frac{21}{64}\sin 3\theta + \frac{63}{128}\sin\theta \\
\sin^{10}\theta &= -\frac{1}{512}\cos 10\theta + \frac{5}{256}\cos 8\theta - \frac{45}{512}\cos 6\theta + \frac{15}{64}\cos 4\theta - \frac{105}{256}\cos 2\theta + \frac{63}{256}
\end{aligned}$$

Table of powers of Sine reduced to Linear combinations

Similarly, here are the formulas for powers of cosine of degree 10 or less.

$$\cos^2\theta = \frac{1}{2}\cos 2\theta + \frac{1}{2}$$
$$\cos^3\theta = \frac{1}{4}\cos 3\theta + \frac{3}{4}\cos\theta$$
$$\cos^4\theta = \frac{1}{8}\cos 4\theta + \frac{1}{2}\cos 2\theta + \frac{3}{8}$$
$$\cos^5\theta = \frac{1}{16}\cos 5\theta + \frac{5}{16}\cos 3\theta + \frac{5}{8}\cos\theta$$
$$\cos^6\theta = \frac{1}{32}\cos 6\theta + \frac{3}{16}\cos 4\theta + \frac{15}{32}\cos 2\theta + \frac{5}{16}$$
$$\cos^7\theta = \frac{1}{64}\cos 7\theta + \frac{7}{64}\cos 5\theta + \frac{21}{64}\cos 3\theta + \frac{35}{64}\cos\theta$$
$$\cos^8\theta = \frac{1}{128}\cos 8\theta + \frac{1}{16}\cos 6\theta + \frac{7}{32}\cos 4\theta + \frac{7}{16}\cos 2\theta + \frac{35}{128}$$
$$\cos^9\theta = \frac{1}{256}\cos 9\theta + \frac{9}{256}\cos 7\theta + \frac{9}{64}\cos 5\theta + \frac{21}{64}\cos 3\theta + \frac{63}{128}\cos\theta$$
$$\cos^{10}\theta = \frac{1}{512}\cos 10\theta + \frac{5}{256}\cos 8\theta + \frac{45}{512}\cos 6\theta + \frac{15}{64}\cos 4\theta + \frac{105}{256}\cos 2\theta + \frac{63}{256}$$