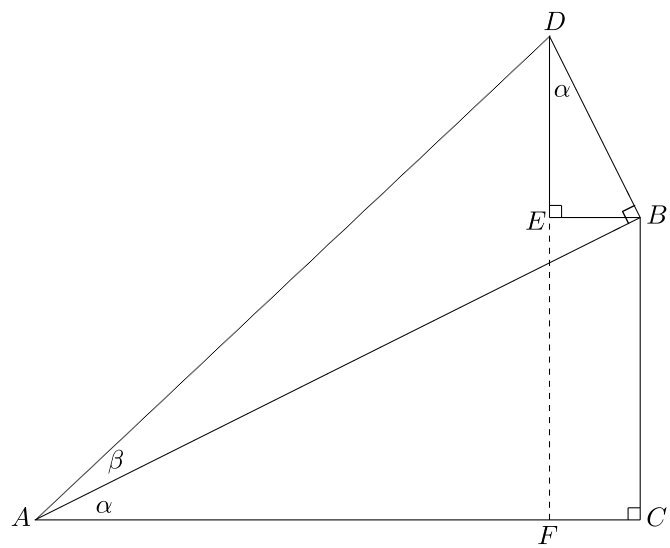
Math 111: Derivation of the Sum Trigonometric Identities

Three particular identities are very important to the study of trigonometry. They are typically know as the sum trigonometric identities. This paper present a geometric proof of the validity of the first two of these identities, along with an algebraic proof of the last one (3).

Theorem 1. *Suppose that* α *and* β *are any two angles. Further suppose that* $\tan\alpha$ *and* $\tan\beta$*, are defined for* α *and* β*. It follows that*

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha, \tag{1}$$
$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta, \text{ and} \tag{2}$$
$$\tan(\alpha+\beta) = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}. \tag{3}$$

Proof. First, draw two triangles on top of each other, with α as the angle of the first triangle, and β as the angle of the second triangle. Label the first triangle as ABC and the second triangle as ABD. The hypotenuse of the first triangle (ABC) is the adjacent leg of the second triangle (ABD). Next, form a new right triangle by dropping an altitude from D down to a point E which lies to the left of B, such that the segment BE is parallel to AC.



Since, BE is parallel to AC, then $\angle ABE$ is the same as $\angle BAC$. Further, since $\angle ABE + \angle EBD = \frac{\pi}{2}$, then $\angle EBD = \frac{\pi}{2} - \alpha$. Finally, since $\angle EBD + \angle EDB = \frac{\pi}{2}$, then $\angle EDB = \alpha$. From $\triangle ABD$,

$$\sin\beta = \frac{BD}{AD} \qquad \cos\beta = \frac{AB}{AD} \tag{4}$$

From $\triangle ABC$,

$$\sin\alpha = \frac{BC}{AB} \qquad \cos\alpha = \frac{AC}{AB}$$
$$\implies BC = AB\sin\alpha \qquad AC = AB\cos\alpha \tag{5}$$

Finally, from $\triangle BDE$,

$$\sin\alpha = \frac{BE}{BD} \qquad \cos\alpha = \frac{DE}{BD}$$
$$\implies BE = BD\sin\alpha \qquad DE = BD\cos\alpha. \tag{6}$$

The sine of $\alpha+\beta$ can be found from $\triangle ADF$ and is

$$\sin(\alpha+\beta) = \frac{DF}{AD} = \frac{BC+DE}{AD}. \tag{7}$$

Using (5), (6), and (4) simplifies (7) to

$$\begin{aligned}\sin(\alpha+\beta) &= \frac{BC+DE}{AD}\\ &= \frac{AB\sin\alpha + BD\cos\alpha}{AD}\\ &= \sin\alpha\frac{AB}{AD} + \frac{BD}{AD}\cos\alpha\\ \sin(\alpha+\beta) &= \sin\alpha\cos\beta + \sin\beta\cos\alpha\end{aligned} \tag{8}$$

Similarly, using (5), (6), and (4) leads to the solution for $\cos(\alpha+\beta)$

$$\begin{aligned}
\cos(\alpha+\beta) &= \frac{AF}{AD} = \frac{AC-CF}{AD} = \frac{AC-BE}{AD} \\
&= \frac{AB\cos\alpha + BD\sin\alpha}{AD} \\
&= \cos\alpha\frac{AB}{AD} + \sin\alpha\frac{BD}{AD} \\
\cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\beta\sin\alpha \qquad (9)
\end{aligned}$$

Finally, to prove (3), divide (8) by (9), and then divide both the top and bottom of the result by $\cos\alpha\cos\beta$ as follows:

$$\begin{aligned}
\tan(\alpha+\beta) &= \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)} = \frac{\sin\alpha\cos\beta + \sin\beta\cos\alpha}{\cos\alpha\cos\beta - \sin\beta\sin\alpha} \\
&= \frac{\frac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta} + \frac{\sin\beta\cos\alpha}{\cos\alpha\cos\beta}}{\frac{\cos\alpha\cos\beta}{\cos\alpha\cos\beta} - \frac{\sin\beta\sin\alpha}{\cos\alpha\cos\beta}} \\
\tan(\alpha+\beta) &= \frac{\tan\alpha + \tan\beta}{1-\tan\alpha\tan\beta} \qquad (10)
\end{aligned}$$

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