Math 111: Solving a Linear Function of sin and cos

When solving Trigonometric Equations, we have so far used identities in our quest. Sometimes the identities will fail. When we have the equation $\boldsymbol{a\sin + b\cos = c}$, we can solve it easily if $c = 0$ (just divide both sides by $\cos\theta$ and using arctan to solve for θ). When $c \neq 0$, then this method will no longer work. In this case, there are at least two method for solving. The first one is not that fun. The second is fun. Let's do the boring one first.

The Shifted Sine Curve

The first method equates the coefficients of our equation with the sine sum of two angles:

$$\begin{aligned} \sin(\theta+\alpha) &= \cos\alpha \quad \cdot \sin\theta \quad + \quad \sin\alpha \quad \cdot\cos\theta \\ &= \quad a \quad \cdot\sin\theta \quad + \quad b \quad \cdot\cos\theta = c \end{aligned}$$

It would be convenient if we could state that $a = \sin\alpha$ and $b = \cos\alpha$, but it's not true for all values of a and b and ignores the relationship between a and b. Rather, we will use the identity $\sin^2\alpha + \cos^2\beta = 1$ to define $\sin\alpha$ and $\cos\alpha$ explicitly. Dividing both sides of the equation by $\sqrt{a^2+b^2}$ gives the appropriate scaling

$$\begin{array}{ccccc} \left(\dfrac{a}{\sqrt{a^2+b^2}}\right)\cdot\sin\theta & + & \left(\dfrac{b}{\sqrt{a^2+b^2}}\right)\cdot\cos\theta & = & \dfrac{c}{\sqrt{a^2+b^2}} \\ \cos\alpha \quad \cdot\sin\theta & + & \sin\alpha \quad \cdot\cos\theta & = & \sin(\theta+\alpha) \end{array}$$

So we solve the equation by first finding α from the two equations on the left side above:

$$\cos\alpha = \frac{a}{\sqrt{a^2+b^2}} \qquad\qquad \sin\alpha = \frac{b}{\sqrt{a^2+b^2}}$$

Note that these answers depend on the sign of a and b. Make sure that α is in the correct quadrant when using arcsin or arccos to solve for α. Finally, we solve the equation on the right side above:

$$\sin(\theta+\alpha) = \frac{c}{\sqrt{a^2+b^2}} \quad \Longrightarrow \quad \theta = -\alpha + \arcsin\left(\frac{c}{\sqrt{a^2+b^2}}\right)$$

Note also that there will be two solutions per period of sin. Thus you could say in general that two solutions are:

$$\theta = -\alpha + \arcsin\left(\frac{c}{\sqrt{a^2+b^2}}\right) \theta = -\alpha + \pi - \arcsin\left(\frac{c}{\sqrt{a^2+b^2}}\right)$$

Other solutions will be multiples of 2π from these.

Second Way: Using $\tan\left(\frac{\theta}{2}\right)$. Turn the page!

TURN THE PAGE!

Second way: Using $\tan\left(\frac{\theta}{2}\right)$.

Using the substitution $t = \tan\left(\frac{\theta}{2}\right)$, we can write any rational function of $\sin\theta$, $\cos\theta$, $\tan\theta$, $\csc\theta$, $\sec\theta$, $\cot\theta$ as a polynomial in t. This allows for a simple solution to $\boldsymbol{a\sin + b\cos = c}$.

Since $t = \tan\left(\frac{\theta}{2}\right)$, it follows that $\sin\left(\frac{\theta}{2}\right) = \frac{t}{\sqrt{1+t^2}}$ and $\cos\left(\frac{\theta}{2}\right) = \frac{1}{\sqrt{1+t^2}}$ Using the double angle formulas, we get

$$\sin\theta = 2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) = 2\left(\frac{t}{\sqrt{1+t^2}}\right)\left(\frac{1}{\sqrt{1+t^2}}\right) = \frac{2t}{1+t^2}$$
$$\cos\theta = \cos^2\left(\frac{\theta}{2}\right) - \sin^2\left(\frac{\theta}{2}\right) = \left(\frac{1}{\sqrt{1+t^2}}\right)^2 - \left(\frac{t}{\sqrt{1+t^2}}\right)^2 = \frac{1-t^2}{1+t^2}$$

It follows that the other four trigonometric functions can be found from these. Thus,

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{2t}{1-t^2} \qquad \sec\theta = \frac{1}{\cos\theta} = \frac{1+t^2}{1-t^2} \qquad \csc\theta = \frac{1}{\sin\theta} = \frac{1+t^2}{2t}$$

Finally, to solve the equation we substitute:

$$c = a\sin\theta + b\cos\theta = a\left[\frac{2t}{1+t^2}\right] + b\left[\frac{1-t^2}{1+t^2}\right]$$

Multiply both sides by $(1+t^2)$ and collecting like terms yields a polynomial of t:

$$c(1+t^2) = 2at + b(1-t^2)$$
$$c + ct^2 = 2at + b - bt^2$$
$$(c+b)t^2 - 2at + c - b = 0$$

This can be solved using the quadratic formula:

$$\tan\left(\frac{\theta}{2}\right) = t = \frac{2a \pm \sqrt{4a^2 - 4(c+b)(c-b)}}{2(c+b)} = \frac{2a \pm 2\sqrt{a^2 - (c^2 - b^2)}}{2(c+b)} = \frac{a \pm \sqrt{a^2 + b^2 - c^2}}{c+b}$$

Thus, the two solutions are

$$\theta = 2\arctan\left(\frac{a \pm \sqrt{a^2 + b^2 - c^2}}{c+b}\right)$$

Note that this method does not work if the solution is the angle π. Thus, we should test the angle π in our original equation to make sure it is or is not a solution.