Math 111: Easy to Memorize Derivation of Trigonometric Identities

Many of the trigonometric identities can be derived in succession from the following identities: (the first and second identities indicate that sin and cos are odd and even functions, respectively. The second two are known as the sum identies.)

Must Memorize!

$$\sin(-\theta) = -\sin\theta, \text{ (odd function)} \tag{1}$$
$$\cos(-\theta) = \cos\theta, \text{ (even function)} \tag{2}$$
$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha, \text{ and} \tag{3}$$
$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta. \tag{4}$$

Suppose that $\beta = -w$, then (3) simplifies to

$$\begin{aligned} \sin(\alpha + (-w)) &= \sin\alpha\cos(-w) + \sin(-w)\cos\alpha && \text{by (3)} \\ &= \sin\alpha\cos w - \sin w\cos\alpha && \text{by (1) and (2)} \end{aligned}$$

Since w is an arbitrary label, then β will do as well. Hence, the sine difference formula is

$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \sin\beta\cos\alpha \tag{5}$$

Similarly, equation (4) simplifies to the cosine difference formula:

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta \tag{6}$$

To find the sum identity for $\tan(\alpha+\beta)$, divide (3) by (4) as follows:

$$\tan(\alpha+\beta) = \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)} = \frac{\sin\alpha\cos\beta + \sin\beta\cos\alpha}{\cos\alpha\cos\beta - \sin\alpha\sin\beta} \tag{7}$$

Divide both the top and bottom of (7) by $\cos\alpha\cos\beta$ results with the simplified formula

$$\tan(\alpha+\beta) = \frac{\dfrac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta} + \dfrac{\sin\beta\cos\alpha}{\cos\alpha\cos\beta}}{\dfrac{\cos\alpha\cos\beta}{\cos\alpha\cos\beta} - \dfrac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}} = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} \tag{8}$$

Because $\tan(-\theta) = \dfrac{\sin(-\theta)}{\cos(-\theta)} = \dfrac{-\sin\theta}{\cos\theta} = -\tan\theta$, then it follows that

$$\tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}. \tag{9}$$

The Double Angle identities can be derived from equations (3) and (4). Suppose $\alpha = \beta = \theta$, then (3) simplifies as

$$\sin(\theta+\theta) = \sin\theta\cos\theta + \sin\theta\cos\theta.$$

Hence,

$$\sin(2\theta) = 2\sin\theta\cos\theta. \tag{10}$$

Similarly,

$$\cos 2\theta = \cos\theta\cos\theta - \sin\theta\sin\theta = \cos^2\theta - \sin^2\theta, \text{ and} \tag{11}$$
$$\tan 2\theta = \frac{\tan\theta + \tan\theta}{1 - \tan\theta\tan\theta} = \frac{2\tan\theta}{1 - \tan^2\theta}. \tag{12}$$

The first of the Pythagorean identities can be found by setting $\alpha = \beta = \theta$ in (6). Hence,

$$\cos(\theta-\theta) = \sin\theta\sin\theta + \cos\theta\cos\theta.$$

So,

$$\begin{aligned} \cos(\theta-\theta) = \cos 0 &= \sin\theta\sin\theta + \cos\theta\cos\theta \\ 1 &= \sin^2\theta + \cos^2\theta. \end{aligned} \tag{13}$$

Dividing both sides of (13) by $\cos^2\theta$ yields

$$\tan^2\theta + 1 = \sec^2\theta. \tag{14}$$

Dividing both sides of (13) by $\sin^2\theta$ yields

$$1 + \cot^2\theta = \csc^2\theta. \tag{15}$$

Equations (11) and (13) can generate the Power Reducting identities. Using $\cos^2\theta = 1 - \sin^2\theta$, (11) can be written as

$$\cos(2\theta) = (1 - \sin^2\theta) - \sin^2\theta = 1 - 2\sin^2\theta.$$

Solving the above equation for $\sin^2\theta$ yields

$$\sin^2\theta = \frac{1-\cos(2\theta)}{2}. \tag{16}$$

Similarly, solving the above equation for $\cos^2\theta$ yields

$$\cos^2\theta = \frac{1+\cos(2\theta)}{2}, \text{ and} \tag{17}$$

$$\tan^2\theta = \frac{\sin^2\theta}{\cos^2\theta} = \frac{1-\cos(2\theta)}{1+\cos(2\theta)}. \tag{18}$$

The product identities can be found using equations (3) through (6). For example, adding (3) and (5) yields

$$\begin{aligned}\sin(\alpha-\beta) + \sin(\alpha+\beta) &= \sin\alpha\cos\beta - \sin\beta\cos\alpha + \sin\alpha\cos\beta + \sin\beta\cos\alpha \\ \sin(\alpha-\beta) + \sin(\alpha+\beta) &= 2\sin\alpha\cos\beta.\end{aligned}$$

Hence,

$$\sin\alpha\cos\beta = \frac{1}{2}\left[\sin(\alpha-\beta) + \sin(\alpha+\beta)\right]. \tag{19}$$

Similarly, adding (4) and (6) together and subtracting (4) from (6) yields:

$$\cos\alpha\sin\beta = \frac{1}{2}\left[\sin(\alpha-\beta) + \sin(\alpha+\beta)\right], \tag{20}$$

$$\cos\alpha\cos\beta = \frac{1}{2}\left[\cos(\alpha-\beta) + \cos(\alpha+\beta)\right], \text{ and} \tag{21}$$

$$\sin\alpha\sin\beta = \frac{1}{2}\left[\cos(\alpha-\beta) - \cos(\alpha+\beta)\right]. \tag{22}$$

Solving $u = \alpha + \beta$ and $v = \alpha - \beta$ for α and β gives $\alpha = \frac{u+v}{2}$ and $\beta = \frac{u-v}{2}$. Plug them into (19) yields:

$$\begin{aligned} & \frac{1}{2}\left[\sin(\alpha-\beta) + \sin(\alpha+\beta)\right] = \sin\alpha\cos\beta \\ \Longrightarrow \quad & \sin(\alpha-\beta) + \sin(\alpha+\beta) = 2\sin\alpha\cos\beta \\ \Longrightarrow \quad & \sin\left(\frac{u+v}{2} - \frac{u-v}{2}\right) + \sin\left(\frac{u+v}{2} + \frac{u-v}{2}\right) = 2\sin\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right) \\ \Longrightarrow \quad & \sin u + \sin v = 2\sin\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right)\end{aligned}$$

Since u and v are arbitrary labels, then α and β will do just as well. Hence,

$$\sin\alpha + \sin\beta = \quad 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) \tag{23}$$

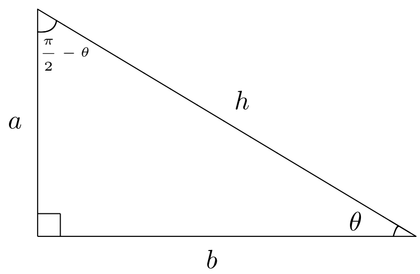
Similarly, replacing α by $\dfrac{\alpha+\beta}{2}$ and β by $\dfrac{\alpha-\beta}{2}$ into (20), (21), and (22) yields

$$\sin\alpha - \sin\beta = \quad 2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right) \tag{24}$$

$$\cos\alpha + \cos\beta = \quad 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) \tag{25}$$

$$\cos\alpha - \cos\beta = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right) \tag{26}$$

Next, the "Co-function Identities" can be derived by using the definition of the functions. Draw a right triangle, label the sides as a, b, and h, where h is the hypotenuse. Label the angles opposite side a and b as θ and B. Note that $\theta + B = \frac{\pi}{2}$, so $B = \frac{\pi}{2} - \theta$. It follows that the each ratios between the sides of the triangle can be written two different ways:



$$\frac{a}{h} = \sin\theta = \cos\left(\frac{\pi}{2} - \theta\right) \qquad \frac{h}{a} = \csc\theta = \sec\left(\frac{\pi}{2} - \theta\right)$$
$$\frac{b}{h} = \cos\theta = \sin\left(\frac{\pi}{2} - \theta\right) \qquad \frac{h}{b} = \sec\theta = \csc\left(\frac{\pi}{2} - \theta\right)$$
$$\frac{a}{b} = \tan\theta = \cot\left(\frac{\pi}{2} - \theta\right) \qquad \frac{b}{a} = \cot\theta = \tan\left(\frac{\pi}{2} - \theta\right)$$

Collected all together, we have equations (27) to (32)

$$\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right) \tag{27}$$
$$\csc\theta = \sec\left(\frac{\pi}{2} - \theta\right) \tag{28}$$
$$\cos\theta = \sin\left(\frac{\pi}{2} - \theta\right) \tag{29}$$
$$\sec\theta = \csc\left(\frac{\pi}{2} - \theta\right) \tag{30}$$
$$\tan\theta = \cot\left(\frac{\pi}{2} - \theta\right) \tag{31}$$
$$\cot\theta = \tan\left(\frac{\pi}{2} - \theta\right) \tag{32}$$
$$\tag{33}$$

The periods of $\sin\theta$ and $\cos\theta$ are 2π, but $\tan\theta$ is π. The reciprocal functions follow that same pattern (where n is an integer):

$$\sin(\theta) = \sin(\theta + 2\pi n) \qquad \csc(\theta) = \csc(\theta + 2\pi n) \tag{34}$$
$$\cos(\theta) = \cos(\theta + 2\pi n) \qquad \sec(\theta) = \sec(\theta + 2\pi n) \tag{35}$$
$$\tan(\theta) = \tan(\theta + \pi n) \qquad \cot(\theta) = \cot(\theta + \pi n) \tag{36}$$

When figuring out the period of any $f(ax + b)$ where a and b are real numbers, it still follows the same pattern: So, for example, $\sin(12\theta + 2)$, the values of θ will vary between:

$$0 \leqslant 12\theta + 2 \leqslant 2\pi$$
$$-2 \leqslant 12\theta \leqslant 2\pi - 2$$
$$\frac{-2}{12} \leqslant \theta \leqslant \frac{2\pi - 2}{12}$$
$$-\frac{1}{6} \leqslant \theta \leqslant \frac{\pi - 1}{6}$$

Thus, you can see that the period of this is

$$P = \frac{\pi - 1}{6} - \frac{-1}{6} = \frac{\pi}{6}.$$

In general, it will be $\frac{\text{period}}{a}$ for $f(ax + b)$. If f is tan or cot, the period is π, and for the others the period will be 2π

Law of Sines, Cosines, and Tangents

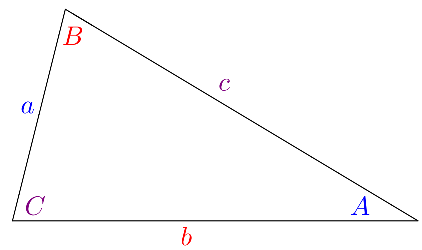


Figure 1: Reference Triangle for Law of Sines, Cosines, and Tangents

The same figure applies to the Law of Sines, the Law of Cosines, and the Law of Tangents. The angles and sides are color coded so you can easily see the relations between them. First, the Law of Sines:

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin B}{c} \tag{37}$$

The Law of Cosines can be put to memory easily by the connection to the Pythagorean Theorem. When any of the angles are a right angle, it simplifies to the Pythagorean Theorem. Thus, start with the Pythagorean theorem, and then fill in the last Cosine part to complete it:

$$a^2 = b^2 + c^2 - 2bc\cos A \tag{38}$$

$$b^2 = a^2 + c^2 - 2ac\cos B \tag{39}$$

$$c^2 = a^2 + b^2 - 2ab\cos C \tag{40}$$

The Law of Tangents states:

$$\frac{a-b}{a+b} = \frac{\tan\frac{1}{2}(A-B)}{\tan\frac{1}{2}(A+B)} \tag{41}$$

$$\frac{a-c}{a+c} = \frac{\tan\frac{1}{2}(A-C)}{\tan\frac{1}{2}(A+C)} \tag{42}$$

$$\frac{b-c}{b+c} = \frac{\tan\frac{1}{2}(B-C)}{\tan\frac{1}{2}(B+C)} \tag{43}$$