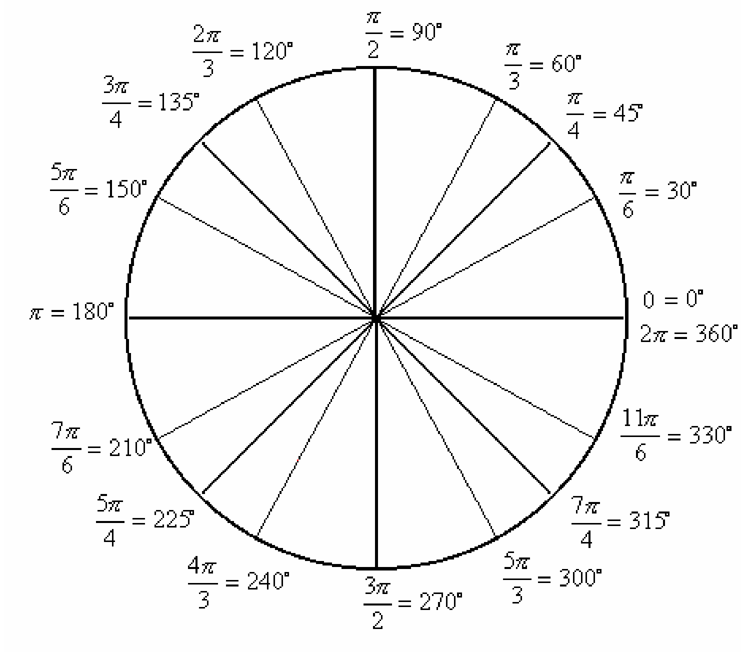
Trigonometry Review with the Unit Circle: All the trig. you'll ever need to know in Calculus

Objectives: This is your review of trigonometry: angles, six trig. functions, identities and formulas, graphs: domain, range and transformations.



Angle Measure

Angles can be measured in 2 ways, in degrees or in radians. The following picture shows the relationship between the two measurements for the most frequently used angles. **Notice**, degrees will always have the degree symbol above their measure, as in "452°", whereas radians are real number without any dimensions, so the number "5" without any symbol represents an angle of 5 radians.

An angle is made up of an initial side (positioned on the positive x-axis) and a terminal side (where the angle lands). It is useful to note the quadrant where the terminal side falls.

Rotation direction

Positive angles start on the positive x-axis and rotate counterclockwise.
Negative angles start on the positive x-axis, also, and rotate clockwise.

Conversion between radians and degrees when radians are given in terms of "π"

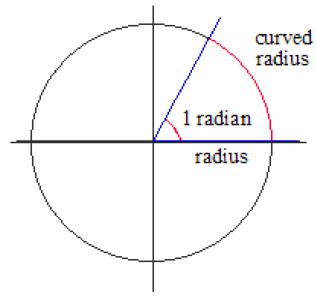
DEGREES → RADIANS: The official formula is $\boxed{\theta^\circ \cdot \frac{\pi}{180^\circ} = \theta \text{ radians}}$

Ex. Convert 120° into radians → **SOLUTION**: $120^\circ \cdot \frac{\pi}{180^\circ} = \frac{2\pi}{3}$ radians

RADIANS → DEGREES: The conversion formula is $\boxed{\theta \text{ radians} \cdot \frac{180^\circ}{\pi} = \theta^\circ}$

Ex. Convert $\frac{\pi}{5}$ into degrees. → **SOLUTION**: $\frac{\pi}{5} \cdot \frac{180^\circ}{\pi} = \frac{180^\circ}{5} = 36^\circ$

For your own reference, $1 \text{ radian} \approx 57.30^{\circ}$



A radian is defined by the radius of a circle.
If you measure off the radius of the circle, then take the straight radius and curved it along the edge of the circle, the angle this arc marks off measures 1 radian.

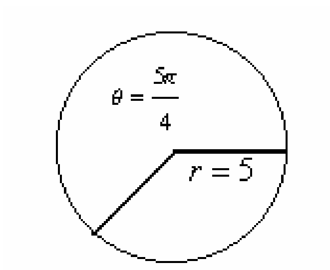
Arc length: when using radians you can determine the arc length of the intercepted arc using this formula:

Arc length = (radius) (degree measure in radians)

OR

$\boxed{s = r\theta}$ There may be times you'll use variations of this formula.

Ex. Find the length of the arc pictured here:



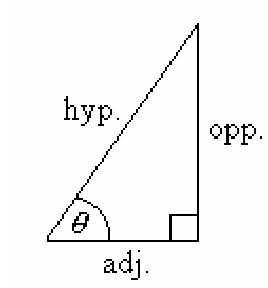
SOLUTION: $s = r\theta$ → you know the values of r and θ

$s = 5 \cdot \frac{5\pi}{4} = \frac{25\pi}{4}$ units for the arc length.

The Trigonometric Ratios

The six trigonometric ratios are defined in the following way based on this right triangle and the angle θ

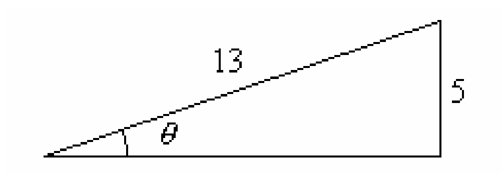
adj. = adjacent side to angle θ
opp. = opposite side to angle θ
hyp. = hypotenuse of the right triangle



SOH CAH TOA → $\sin\theta = \frac{\text{opp.}}{\text{hyp.}}$ $\cos\theta = \frac{\text{adj.}}{\text{hyp.}}$ $\tan\theta = \frac{\text{opp.}}{\text{adj.}}$

Reciprocal functions → $\csc\theta = \frac{\text{hyp.}}{\text{opp.}}$ $\sec\theta = \frac{\text{hyp.}}{\text{adj.}}$ $\cot\theta = \frac{\text{adj.}}{\text{opp.}}$

Ex. Find the exact values of all 6 trigonometric functions of the angle θ shown in the figure.



SOLUTION: first you'll need to determine the 3rd side using $a^2 + b^2 = c^2$ → $a^2 + 5^2 = 13^2$ → $a = 12$
So for the angle labeled θ, ADJACENT = 12, OPPOSITE = 5 and HYPOTENUSE = 13

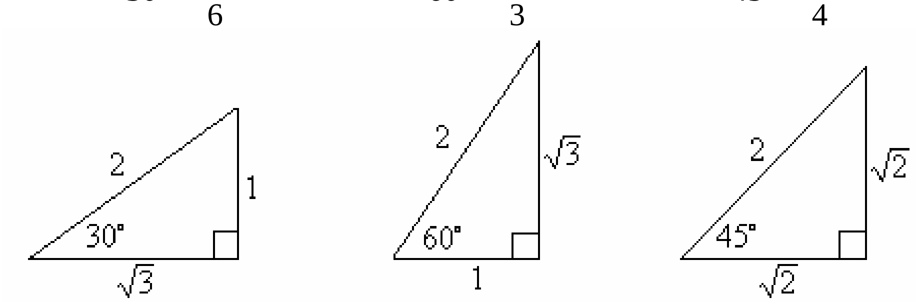
$$\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{5}{13} \qquad \cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{12}{13} \qquad \tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{5}{12}$$

$$\csc\theta = \frac{\text{hyp}}{\text{opp}} = \frac{13}{5} \qquad \sec\theta = \frac{\text{hyp}}{\text{adj}} = \frac{13}{12} \qquad \cot\theta = \frac{\text{adj}}{\text{opp}} = \frac{12}{5}$$

Special Angles

The following triangles will help you to memorize the trig functions of these special angles

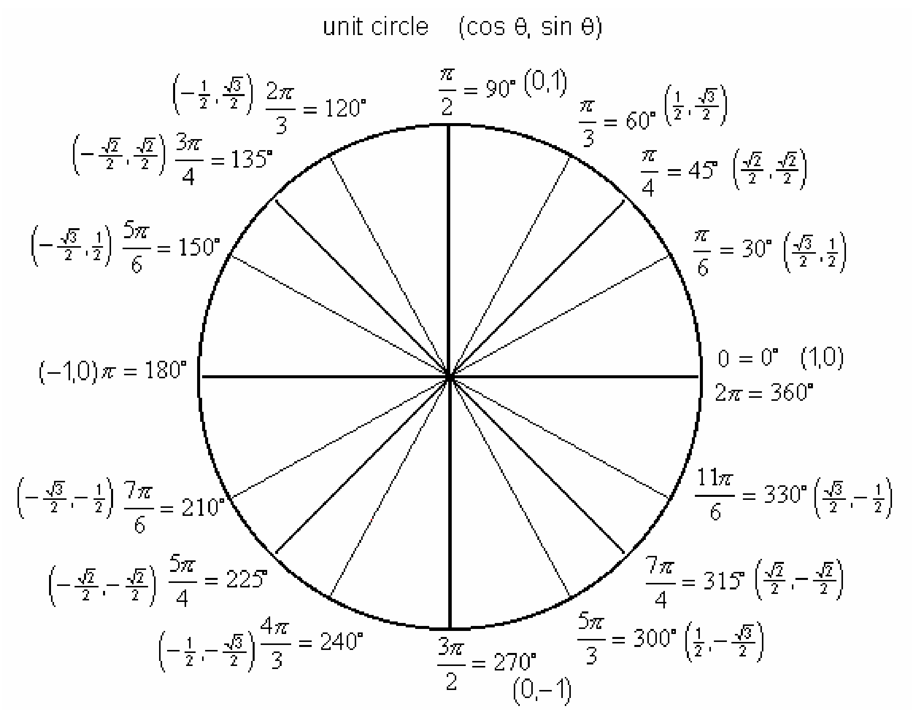
$30^\circ = \frac{\pi}{6}$ $60^\circ = \frac{\pi}{3}$ $45^\circ = \frac{\pi}{4}$



If the triangles are not your preferred way of memorizing exact trig. ratios, then use this table.

θ°	θ^{RAD}	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\csc\theta$	$\sec\theta$	$\cot\theta$
30°	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$	$\sqrt{3}$
45°	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	$\sqrt{2}$	$\sqrt{2}$	1
60°	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2	$\frac{\sqrt{3}}{3}$

… but even better than this is the **unit circle**.



The trig. ratios are defined as …

$$\sin t = y$$

$$\cos t = x$$

$$\tan t = \frac{y}{x}, x \neq 0$$

$$\csc t = \frac{1}{y},\ y \neq 0$$

$$\sec t = \frac{1}{x},\ x \neq 0$$

$$\cot t = \frac{x}{y},\ y \neq 0$$

Domain and Period of Sine and Cosine

Considering the trigonometric ratios as functions where the INPUT values of t come from values (angles) on the unit circle, then you can say the domain of these functions would be all real numbers.

Domain of Sine and Cosine: All real numbers

Based on the way the domain values can start to "cycle" back over the same points to produce the same OUTPUT over and over again, the range is said be periodic. But still, the largest value that sine and cosine OUTPUT on the unit circle is the value of 1, the lowest value of OUTPUT is –1.

Range of Sine and Cosine: [– 1 , 1]

Since the real line can wrap around the unit circle an infinite number of times, we can extend the domain values of t outside the interval $[0,\ 2\pi]$. As the line wraps around further, certain points will overlap on the same (x, y) coordinates on the unit circle. Specifically for the functions sine and cosine, for any value $\sin t$ and $\cos t$ if we add 2π to t we end up at the same (x, y) point on the unit circle.

Thus $\sin(t + n \cdot 2\pi) = \sin t$ and $\cos(t + n \cdot 2\pi) = \cos t$ for any integer n multiple of 2π.

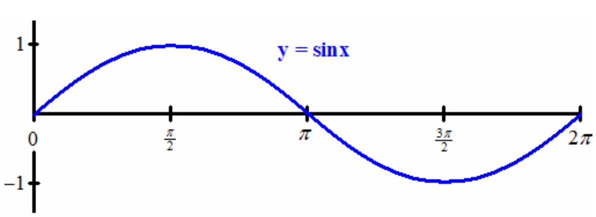
These cyclic natures of the sine and cosine functions make them **periodic functions**.

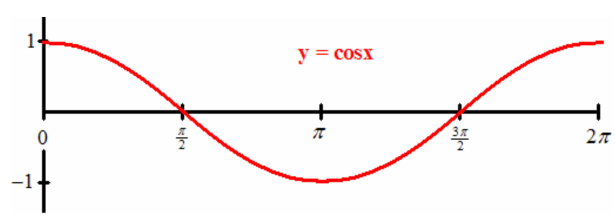
Graphs of Sine and Cosine

Below is a table of values, similar to the tables we've used before. We're going to start thinking of how to get the graphs of the functions $y = \sin x$ and $y = \cos x$.

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{3\pi}{2}$	2π
$y = \sin x$	0	0.5	$\frac{\sqrt{2}}{2} \approx 0.7071$	$\frac{\sqrt{3}}{2} \approx 0.8660$	1	$\frac{\sqrt{2}}{2} \approx 0.7071$	0	–1	0
$y = \cos x$	1	$\frac{\sqrt{3}}{2} \approx 0.8660$	$\frac{\sqrt{2}}{2} \approx 0.7071$	0.5	0	$-\frac{\sqrt{2}}{2} \approx -0.7071$	–1	0	1

Now, if you plot these y-values over the x-values we have from the unwrapped unit circle, we get these graphs.





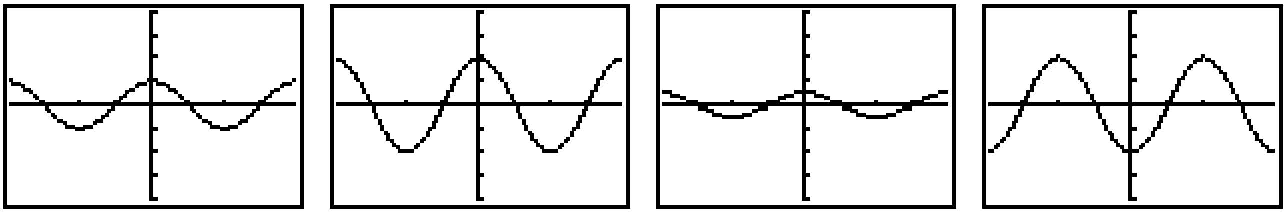
One very misleading fact about these pictures is the domain of the function … remember that the functions of sine and cosine are periodic and they exist for input outside the interval $[0, 2\pi]$. The domain of these functions is all real numbers and these graphs continue to the left and right in the same **sinusoidal** pattern. The range is $[-1,1]$.

Amplitude

When the sine or cosine function has a coefficient in front, such as the value of a in the equation $y = a\sin x$ or $y = a\cos x$, this causes the graph to stretch or shrink its y-values. This is referred to as the **amplitude**.

Ex. Compare the graphs of

$y=\cos x$ | $y = 2\cos x$ | $y=\frac{1}{2}\cos x$ | $y=-2\cos x$



amplitude = 1 | amplitude = 2 | amplitude = ½ | amplitude = 2 with reflection in x-axis

Amplitude is an absolute value quantity. When the coefficient is negative, this causes an x-axis reflection.

Period

If there is a coefficient within the argument in front of the x, this will change the length of the function's **period**. The usual cycle for sine and cosine is on the interval $0 \le x \le 2\pi$, but here's how this can change …

→ Let b be a positive real number. The period of $y = a\sin bx$ and $y = a\cos bx$ is found this way:

$$0 \le bx \le 2\pi \;\rightarrow\; \text{divide by } b \;\rightarrow\; 0 \le x \le \frac{2\pi}{b}$$

NOTE:

If $b > 1$, this will cause the graph to shrink horizontally because the period will be less than 2π.

If $0 < b < 1$, the graph will stretch horizontally making the period greater than 2π.

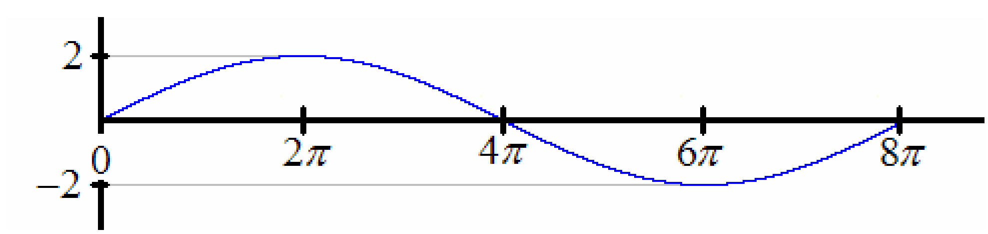
You'll need to adjust the key points of the graph when the period changes! Key points are found by dividing the period length into 4 increments.

Ex. Sketch a graph of $y = 2\sin\left(\frac{1}{4}x\right)$ by hand.

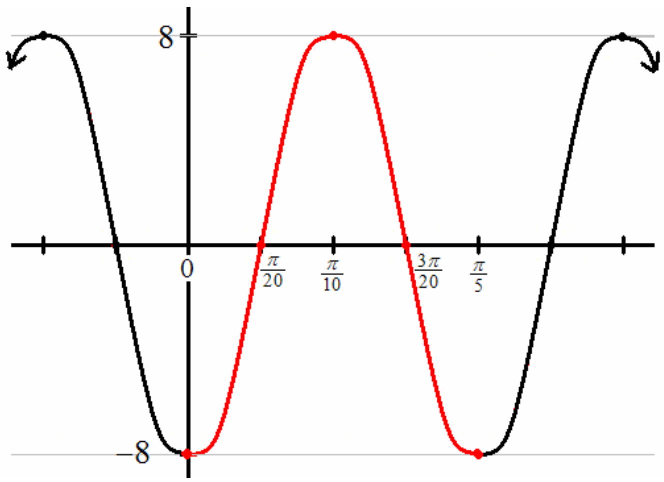
SOLUTION: The coefficient $a = 2$ will effect the range of the graph. The amplitude is 2. Now the period is determined by taking the argument expression inside the function and solving this inequality …

$$0 \le \frac{1}{4}x \le 2\pi \;\rightarrow\; \text{multiply by 4 on all sides} \;\rightarrow\; 0 \le x \le 8\pi$$

You should divide the interval $0 \le x \le 8\pi$ into 4 equal increments.



Ex. Sketch the graph of $y = -8\cos(10x)$.



SOLUTION: The amplitude here is 8.
The negative sign means the graph of cosine will be reflected in the x-axis.

The period for the graph will be

→ $0 \le 10x \le 2\pi$ → $0 \le x \le \frac{\pi}{5}$

The period here is so much smaller than usual, when you graph it on the calculator, it looks too narrow. You'll need to scale the graph down so you can get an accurate picture of the wave.

One full period of this graph is shown in red above. Don't forget, just because you're only graphing one full cycle of the function doesn't mean it "stops" there … these graphs continue on in a periodic motion.

Translations or Phase Shift

For the equation $y = a\sin(bx + c)$ and $y = a\cos(bx + c)$ you can determine the "phase shift" in a way similar to determining the period of the function.

Set up an inequality → $0 \le bx + c \le 2\pi$ → solve for x → $\frac{-c}{b} \le x \le \frac{2\pi - c}{b}$

This new interval represents where the usual cycle for the sine or cosine graph gets shifted to on the x-axis.

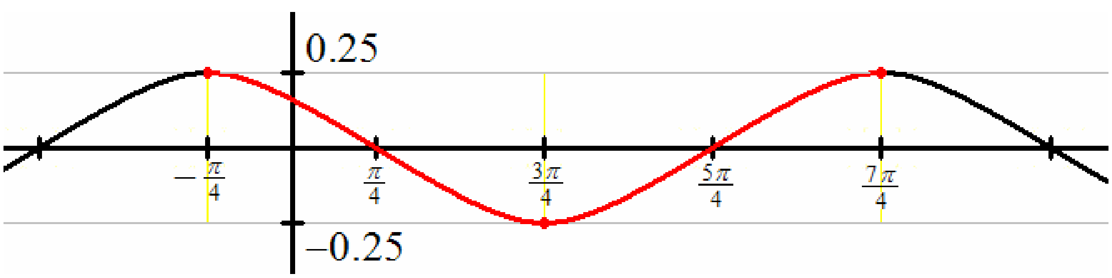
Ex. Sketch the function $y = 0.25\cos\left(x + \frac{\pi}{4}\right)$.

SOLUTION:
Amplitude = 0.25, Period = 2π, now determine the phase shift interval.

$0 \le x + \frac{\pi}{4} \le 2\pi$ → subtract $\frac{\pi}{4}$ → $-\frac{\pi}{4} \le x \le 2\pi - \frac{\pi}{4}$ → $-\frac{\pi}{4} \le x \le \frac{7\pi}{4}$

So, one full cycle of this function's graph will be on the interval $-\frac{\pi}{4} \le x \le \frac{7\pi}{4}$.

After you determine the interval for the phase shift, I recommend labeling the x-axis first with all the critical points. Don't position the y-axis until you've labeled all the points first, then you can decide where the y-axis should fall. The critical points are at $x = -\frac{\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$ and $\frac{7\pi}{4}$

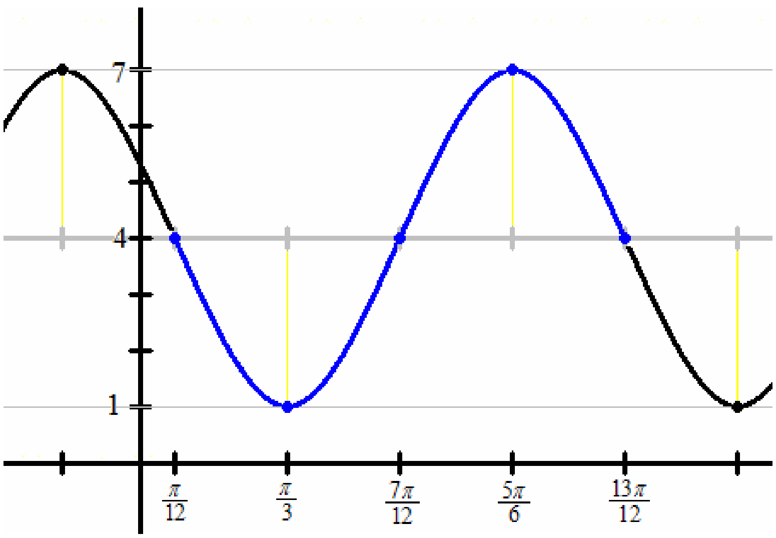


Ex. Graph the function $f(x) = 4 - 3\sin(2x - \frac{\pi}{6})$

SOLUTION: This one's got it all! Amplitude = 3 with a reflection, period = $\frac{2\pi}{b} = \frac{2\pi}{2} = \pi$,

phase shift → $0 \le 2x - \frac{\pi}{6} \le 2\pi$ → $\frac{\pi}{6} \le 2x \le 2\pi + \frac{\pi}{6}$ → $\frac{\pi}{12} \le x \le \frac{13\pi}{12}$

… AND a vertical shift by 4 units. The vertical shift is easy to manage, just prepare the function as you would normally, but shift the x-axis portion of the graph up 4 units in the y-direction.



On these vertically shifted problems, it may help to draw in a "dotted" x-axis to help determine your critical points and sketch the graph. Then when you determine where the y-axis falls, you can draw a solid x-axis where it should go.

Identities and Formulas

Here's a listing of some of the various formulas and identities from trig. which we'll use through calculus.

Reciprocal Identities $\sin u = \frac{1}{\csc u}$ $\cos u = \frac{1}{\sec u}$ $\tan u = \frac{1}{\cot u}$ $\csc u = \frac{1}{\sin u}$ $\sec u = \frac{1}{\cos u}$ $\cot u = \frac{1}{\tan u}$	**Quotient Identities** $\tan u = \frac{\sin u}{\cos u}$ $\cot u = \frac{\cos u}{\sin u}$
Pythagorean Identities $\boxed{\sin^2 u + \cos^2 u = 1}$ also … $\cos^2 u = 1 - \sin^2 u$ and $\sin^2 u = 1 - \cos^2 u$ $\boxed{1 + \tan^2 u = \sec^2 u}$ also … $\tan^2 u = \sec^2 u - 1$ and $1 = \sec^2 u - \tan^2 u$ $\boxed{1 + \cot^2 u = \csc^2 u}$ also … $\cot^2 u = \csc^2 u - 1$ and $1 = \csc^2 u - \cot^2 u$	**Even / Odd Identities** **ODDS** $\sin(-u) = -\sin u$ $\csc(-u) = -\csc u$ $\tan(-u) = -\tan u$ $\cot(-u) = -\cot u$ **EVENS** $\cos(-u) = \cos u$ $\sec(-u) = \sec u$

These formulas will be used a little less often, but when you need them they are located inside your book cover. You will not have to memorize these.

Angle Sum and Difference Formulas

$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta \qquad \cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$$

$$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta \qquad \cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$$

$$\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta} \qquad \tan(\alpha-\beta)=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$$

The Double Angle Formulas

For Sine

$$\sin 2u = 2\sin u\cos u$$

For Cosine

$$\cos 2u=\cos^2 u-\sin^2 u$$

$$\cos 2u = 2\cos^2 u-1$$

$$\cos 2u = 1-2\sin^2 u$$

For Tangent

$$\tan 2u=\frac{2\tan u}{1-\tan^2 u}$$

Power Reduction (a.k.a. Half-Angle) Formulas

$$\sin^2 x=\frac{1-\cos 2x}{2} \qquad \cos^2 x=\frac{1+\cos 2x}{2}$$

The Product to Sum formulas

$$\sin\alpha\cos\beta=\tfrac{1}{2}\left[\sin\left(\alpha+\beta\right)+\sin\left(\alpha-\beta\right)\right]$$

$$\cos\alpha\cos\beta=\tfrac{1}{2}\left[\cos\left(\alpha+\beta\right)+\cos\left(\alpha-\beta\right)\right]$$

$$\sin\alpha\sin\beta=\tfrac{1}{2}\left[\cos\left(\alpha-\beta\right)-\cos\left(\alpha+\beta\right)\right]$$

Sum to Product Formulas

$$\sin x+\sin y=2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

$$\sin x-\sin y=2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$

$$\cos x+\cos y=2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

$$\cos x-\cos y=-2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$

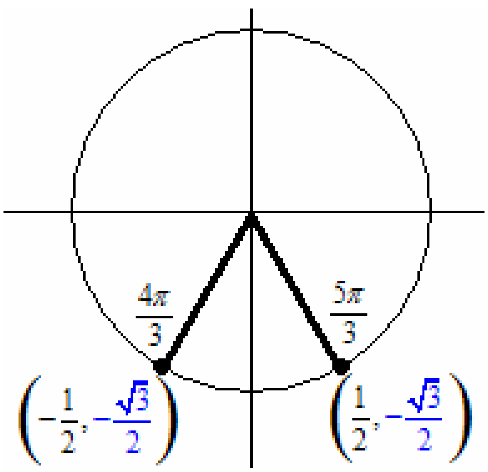
Trigonometric Equations
We'll have to solve some trigonometric equations incidentally throughout calculus. Here's a quick review on how the unit circle can help solve these equations.

Ex) Solve the equation $\sqrt{3}\cdot\csc x + 2 = 0$ on the interval $[0, 2\pi)$.
SOLUTION:

First: Solve for the trig function involved →

$$\sqrt{3}\cdot\csc x + 2 = 0$$
$$\sqrt{3}\cdot\csc x = -2$$
$$\frac{\sqrt{3}\cdot\csc x}{\sqrt{3}} = -\frac{2}{\sqrt{3}}$$
$$\csc x = -\frac{2}{\sqrt{3}}$$

If $\csc x = -\frac{2}{\sqrt{3}}$ then this also means $\sin x = -\frac{\sqrt{3}}{2}$ ← this will help you identify the angles on the unit circle.



Second: Identify the angles which satisfy the equation:

Sine is defined by as the y-coordinates on the unit circle. You're looking for the unit circle angles where the y-coordinate is $-\sqrt{3}/2$.

This happens in two places $x = \frac{4\pi}{3}$ and $x = \frac{5\pi}{3}$.

The solutions are $x = \frac{4\pi}{3}$ and $x = \frac{5\pi}{3}$.