Math 112

Hypothetical Exam 1, (Chapter 1, 2 in Thomas, Weir, Hass, and Giordano, 11th Ed.)
November 5, 1955
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Show all your work to receive credit. All answers must be justified to get full credit.

These questions are intended to give students in Math 112 some idea of the types of questions which could be asked on an exam. The questions may not cover all of the topics which will be on your exam (and they may cover more topics than are on your exam). The length of your exam may be shorter than this practice exam. **Working these problems is not a substitute for studying your notes, reading the book, or doing homework problems.**

Fill in the Blank

Write the best answer in the spaces provided

1. (2 pts) A function $y = f(x)$ that is even is symmetric with respect to the y-axis.
2. (2 pts) A function $y = f(x)$ that is odd is symmetric with respect to the origin.
3. (2 pts) A function $y = f(x)$ is increasing if it rises as you move from left to right.
4. (2 pts each) Write down next to each of the following problems the effect the transformation has on the graph of $y = f(x)$.

   (a) $y = f(x) + 5$

   moves the graph vertically 5 units.

   (b) $y = -f(x)$

   Reflects graph over the x-axis

   (c) $y = 5 + 2f(x + 3)$

   Moves the graph to the left 3 units
   stretches by a factor of 2 vertically
   then shifts graph vertically up 5 units.

   (d) $y = f(x - 4)$

   Moves graph 4 units to the right.

   (e) $y = f(-x)$

   Reflects graph over the y-axis

5. (2 pts each) Fill in the blank for the answers for the following limits based on the graph below. Each unit on the graph corresponds to one unit.

(a) $\lim_{x \to -1} f(x) =$ dne (Note, $\lim_{x \to -1^+} f(x) = 3$)

(b) $\lim_{x \to 10^-} f(x) =$ 9

(c) $\lim_{x \to 17^+} f(x) =$ 2

(d) $\lim_{x \to 4^+} f(x) =$ 6

(e) $\lim_{x \to 10^+} f(x) =$ 9

(f) $\lim_{x \to 17} f(x) =$ dne

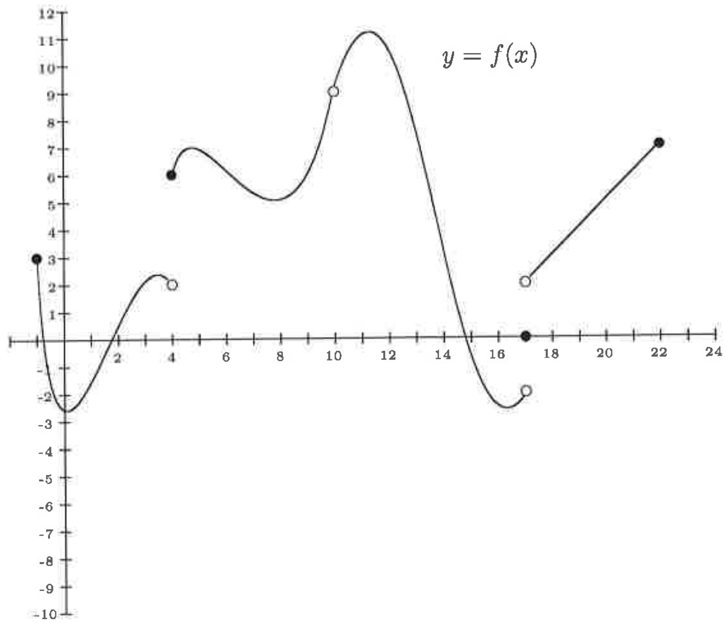
(g) $\lim_{x \to 10} f(x) =$ 9

(h) $\lim_{x \to 4} f(x) =$ dne

(i) $f(10) =$ dne

(j) $f(4) =$ 6

(k) $f(17) =$ 0



(l) At which points is $f(x)$ continuous?

Everywhere except $x=4$, $x=10$, $x=17$

Interval ⇒ $[-1,4) \cup [4,10) \cup (10,17) \cup (17,22)$

(m) At which points is $f(x)$ discontinuous?

$(-\infty,-1)$ and $(22,\infty)$

and at $x = 4, 10, 17$

(n) At which points is $f(x)$ only right continuous?

$x = -1, 4$

(o) At which points is $f(x)$ only left continuous?

$x = 22$

Show Your Work
Show all work clearly and neatly. No work shown means no credit will be given. Use correct notation to get full credit. Reserve scratch paper work for scratch paper, which means only include necessary work on the exam. Erase all mistakes neatly. Keep it neat!

6. (5 pts) Find an equation for the circle with the center $(-1, 5)$ and radius $\sqrt{10}$.

$(x-h)^2 + (y-k)^2 = r^2$ is the standard equation with center (h,k) and radius r.

So $\boxed{(x+1)^2 + (y-5)^2 = 10}$

7. (5 pts) Using the trigonometric identity $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$, find $\cos\left(\frac{\pi}{12}\right)$. (Hint: $\frac{\pi}{12} = \frac{\pi}{3} - \frac{\pi}{4}$).

$$\cos\left(\frac{\pi}{12}\right) = \cos\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \cos\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{4}\right) + \sin\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{4}\right)$$

$$= \frac{1}{2}\left(\frac{\sqrt{2}}{2}\right) + \frac{\sqrt{3}}{2}\left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{2}\left(\frac{1}{2} + \frac{\sqrt{3}}{2}\right)$$

$$= \frac{\sqrt{2}}{2}\frac{1}{2}(1+\sqrt{3}) = \frac{\sqrt{2}}{4}(1+\sqrt{3}) = \frac{\sqrt{2}(1+\sqrt{3})}{4}$$

$$= \boxed{\frac{\sqrt{2}+\sqrt{6}}{4}}$$

8. (5 pts) Calculate $\lim_{x\to 0^+} \frac{\sin x}{x}$ by using the sandwich theorem. (Hint: use the inequality that $x - \frac{x^2}{6} < \sin x < x$ for the interval $0 < x < \frac{\pi}{2}$).

So $$x - \frac{x^2}{6} < \sin x < x \quad (\text{when } 0 < x < \tfrac{\pi}{2})$$

Divide by x $$1 - \frac{x}{6} < \frac{\sin x}{x} < 1$$

Take limit of all sides $$\lim_{x\to 0^+} 1 - \frac{x}{6} < \lim_{x\to 0^+} \frac{\sin x}{x} < \lim_{x\to 0^+} 1$$

$$1 < \lim_{x\to 0^+} \frac{\sin x}{x} < 1$$

Thus, by the sandwich theorem

$$\boxed{\lim_{x\to 0^+} \frac{\sin x}{x} = 1}$$

9. (5 pts) Compute the following limit:

$$\lim_{x \to 2} \frac{\sqrt{x^2+12}-4}{x-2}$$

Since when $x=2$, we get $\frac{\sqrt{4+12}-4}{2-2} = \frac{0}{0}$, then we need to multiply by conjugate of top. So.

$$= \lim_{x \to 2} \frac{\sqrt{x^2+12}-4}{x-2} \frac{(\sqrt{x^2+12}+4)}{(\sqrt{x^2+12}+4)} = \lim_{x \to 2} \frac{x^2+12-16}{(x-2)[\sqrt{x^2+12}+4]}$$

$$= \lim_{x \to 2} \frac{x^2-4}{(x-2)(\sqrt{x^2+12}+4)} = \lim_{x \to 2} \frac{(x+2)\cancel{(x-2)}}{\cancel{(x-2)}(\sqrt{x^2+12}+4)}$$

$$= \lim_{x \to 2} \frac{x+2}{\sqrt{x^2+12}+4} = \frac{2+2}{\sqrt{4+12}+4} = \frac{4}{4+4} = \boxed{\frac{1}{2}}$$

10. (5 pts) Calculate $\lim_{x \to -1^-} \left(\frac{x^2}{2} - \frac{1}{x}\right)$.

$$= \frac{(-1)^2}{2} - \left(\frac{1}{-1}\right) = \frac{1}{2} + 1 = \boxed{\frac{3}{2}}$$

11. (5 pts) Show that $\lim_{x\to 0} \frac{\sin 5x}{\sin 4x} = \frac{5}{4}$.

Since $\lim_{x\to 0} \frac{\sin x}{x} = 1$ (from problem 8), then

$$\lim_{x\to 0} \frac{\sin 5x}{\sin 4x} = \lim_{x\to 0} \frac{\sin 5x}{5x} \frac{4x}{\sin 4x} \frac{5x}{4x}$$

$$= \lim_{x\to 0} \frac{\sin 5x}{5x} \cdot \lim_{x\to 0} \frac{4x}{\sin 4x} \cdot \lim_{x\to 0} \frac{5x}{4x}$$

$u = 5x$ $\quad$ $w = 4x$

$$= \lim_{u\to 0} \frac{\sin u}{u} \cdot \lim_{w\to 0} \frac{1}{\frac{\sin w}{w}} \cdot \lim_{x\to 0} \frac{5}{4}$$

$$= (1) \cdot \frac{1}{(1)} \cdot \frac{5}{4}$$

$$= \boxed{\frac{5}{4}}$$

12. (10 pts) Find the equation for the tangent line to the curve $f(x) = 4 - x^2$ at the point $(2, 3)$.

We need to slope. That's the derivative

So $f'(x) = -2x \Rightarrow f'(2) = -2(2) = -4 = m$

Thus, the slope is -4.

Next,

$$y = -4x + b$$

Use the point $(2,3)$ to find b.

Thus,

$$3 = -4(2) + b$$

$$3 = -8 + b$$

$$b = 11$$

So

$$\boxed{y = -4x + 11}$$

13. (7 pts) Compute the derivative of $f(x) = 2\sqrt{x}$ at the point $x = 4$.

Using the definition gives.

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

So

$$f'(4) = \lim_{h \to 0} \frac{f(4+h) - f(4)}{h} = \lim_{h \to 0} \frac{2\sqrt{4+h} - 2\sqrt{4}}{h}$$

$$= \lim_{h \to 0} \left(\frac{2\sqrt{4+h} - 4}{h}\right) \frac{(2\sqrt{4+h} + 4)}{(2\sqrt{4+h} + 4)}$$

$$= \lim_{h \to 0} \frac{4(4+h) - 16}{h(2\sqrt{4+h} + 4)} = \lim_{h \to 0} \frac{4h}{h(2\sqrt{4+h} + 4)}$$

$$= \lim_{h \to 0} \frac{4}{2\sqrt{4+h} + 4} = \frac{4}{2\sqrt{4} + 4} = \frac{4}{4+4} = \boxed{\frac{1}{2}}$$

14. (5 pts) Prove $\lim_{x \to -2} x^2 = 4$ by using the precise definition of the limit.

The precise definition is: If $|x-a| < \delta$, then $|f(x) - L| < \varepsilon$ (*)

So we first choose $\varepsilon > 0$. Then we need to pick a δ that depends on ε such that the statement follows.

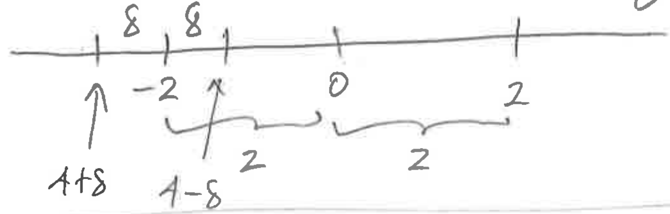
Scratch work.

$|f(x) - L| = |x^2 - 4| = |(x+2)(x-2)| = |x+2||x-2|$

Note: We will need $|x+2|$ since that's the left side of (*)

So if $|x+2| < \delta$, then

$4 - \delta < |x-2| < 4 + \delta$



Suppose that $\delta \le 1$. It follows $4 - 1 < |x-2| < 4 + 1$

$3 < |x-2| < 5$

Choose $\delta = \min\{1, \frac{\varepsilon}{5}\}$

It follows that

$0 < |x+2| < \delta \Rightarrow |(x+2)(x-2)| < 5\left(\frac{\varepsilon}{5}\right) = \varepsilon$

Thus $|x^2 - 4| < \varepsilon$, which means

that $\lim_{x \to -2} x^2 = 4$