Math 212

Hypothetical Exam 4, (Chapter 5 in Thomas, Weir, Hass, and Giordano, 11th Ed.)
October 30, 1961
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Show all your work to receive credit. All answers must be justified to get full credit.

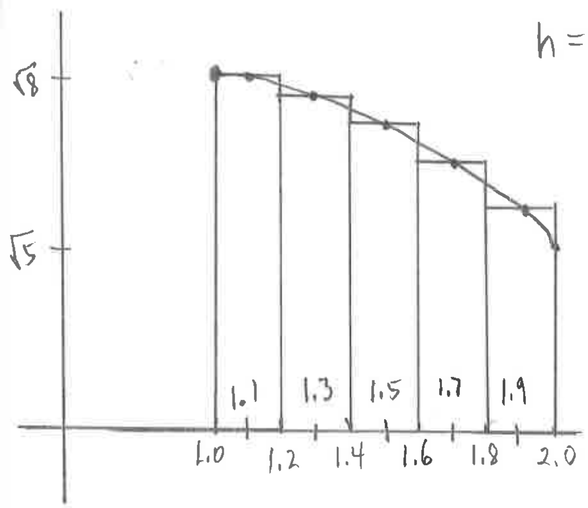
These questions are intended to give students in Math 212 some idea of the types of questions which could be asked on an exam. The questions may not cover all of the topics which will be on your exam (and they may cover more topics than are on your exam). The length of your exam may be shorter than this practice exam. **Working these problems is <u>not</u> a substitute for studying your notes, reading the book, or doing homework problems.**

Show Your Work
Show all work clearly and neatly. No work shown means no credit will be given. Use correct notation to get full credit. Reserve scratch paper work for scratch paper, which means only include necessary work on the exam. Erase all mistakes neatly. Keep it neat!

1. (10 pts) Using rectangles whose height is given by the value of the function at the midpoint of the rectangle's base (the midpoint rule), estimate the area under the graph of the function

$$f(x) = \sqrt{9 - x^2}$$

between the values of $x = 1$ and $x = 2$. Use 5 rectangles.



$h = \frac{2-1}{5} = .2 \quad x_i = 1 + hi - \frac{h}{2} = 1 + h(i - \frac{1}{2})$

$M = h \sum_{K=1}^{5} f(x_i)$

$= .2\left[f(1.1) + f(1.3) + f(1.5) + f(1.7) + f(1.9)\right]$

$= .2\left[\sqrt{9-1.1^2} + \sqrt{9-1.3^2} + \cdots + \sqrt{9-1.9^2}\right]$

$= .2\left[\sqrt{7.79} + \sqrt{7.31} + \sqrt{6.75} + \sqrt{6.11} + \sqrt{5.39}\right]$

$= .2\left[12.8863132978\right]$

$= \boxed{2.57726265956}$

2.5763627756

compare to actual

2. (10 pts) Write the sum $\sum_{k=1}^{5} k(3k+5)$ without sigma notation. Then evaluate the sum.

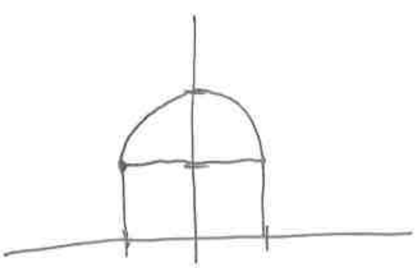
$= 1(3\cdot1+5) + 2(3\cdot2+5) + 3(3\cdot3+5) + 4(3\cdot4+5) + 5(3\cdot5+5)$

$= 8 + 22 + 42 + 68 + 100$

$= 240$

3. (10 pts) Graph the integrand and use areas to evaluate the integral

$$\int_{-1}^{1} 1 + \sqrt{1-x^2}\, dx$$



It's a semicircle w/radius 1 and a rectangle with sides of 1 & 2.

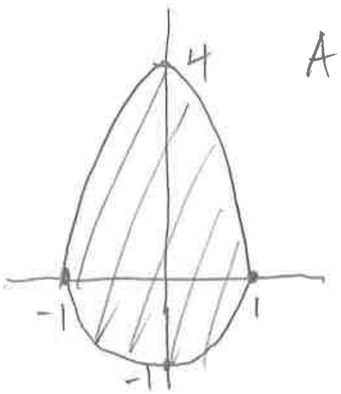
So $\int_{-1}^{1} 1 + \sqrt{1-x^2}\, dx = \pi(1)^2 + 2(1)$

$= \boxed{\pi + 2}$

4. (10 pts) Express $\lim_{||P||\to 0} \sum_{k=1}^{n} (c_k^2 - 3c_k)\Delta x_k$ as a definite integral, where P is a partition of $[-7, 5]$

$\int_{-7}^{5} x^2 - 3x \; dx$

5. (10 pts) Find the area of the region enclosed by the curves $4x^2 + y = 4$ and $x^4 - y = 1$.



$$A = 2\int_0^1 4 - 4x^2 - (x^4 - 1)\, dx$$

$$= 2\int_0^1 5 - 4x^2 - x^4\, dx = 2\left[5x - \frac{4}{3}x^3 - \frac{1}{5}x^5\right]_0^1$$

$$= 2\left[5 - \frac{4}{3} - \frac{1}{5}\right] = \frac{104}{15} = 6.9333333$$

6. Evaluate the following integrals:

(a) (10 pts) $\int_{-2}^{0} 2x + 5\, dx$ $= \left. x^2 + 5x \right]_{-2}^{0} = 0 - \left((-2)^2 + 5(-2)\right)$

$$= -(4 - 10)$$

$$= \boxed{6}$$

(b) (10 pts) $\int_{-\sqrt{3}}^{\sqrt{3}} (t+1)(t^2+4)\, dt$ $= \underbrace{\int_{-\sqrt{3}}^{\sqrt{3}} t(t^2+4)\, dt}_{\text{odd function integrates to 0.}} + \underbrace{\int_{-\sqrt{3}}^{\sqrt{3}} t^2 + 4\, dt}_{\text{even function integrates to } 2\int_0^{\sqrt{3}}}$

$$= 2\int_0^{\sqrt{3}} t^2 + 4\, dt = 2\left(\frac{t^3}{3} + 4t\right)_0^{\sqrt{3}} = 2\left[\frac{3^{3/2}}{3} + 4\sqrt{3}\right]$$

$$= 2\left[\sqrt{3} + 4\sqrt{5}\right] = \boxed{10\sqrt{3}}$$

(c) (10 pts) $\int \tan^7\left(\frac{x}{2}\right)\sec^2\left(\frac{x}{2}\right)\,dx$

$u = \tan\left(\frac{x}{2}\right)$

$du = \frac{1}{2}\sec^2\left(\frac{x}{2}\right)dx$

$$= 2\int u^7\,du = 2\,\frac{u^8}{8} + C$$

$$= \frac{1}{4}u^8 + C = \boxed{\frac{1}{4}\tan^8\left(\frac{x}{2}\right) + C}$$

(d) (10 pts) $\int 3y\sqrt{7-4y^2}\,dy$

$u = 7-4y^2$

$du = -8y\,dy$

$dy = \frac{du}{-8y}$

$$= \int 3\cancel{y}\sqrt{u}\,\frac{du}{-8\cancel{y}}$$

$$= -\frac{3}{8}\int u^{1/2}\,du = -\frac{3}{8}\,\frac{u^{3/2}}{3/2} + C$$

$$= -\frac{\cancel{3}}{8}\cdot\frac{2}{\cancel{3}}\,u^{3/2} + C$$

$$= \boxed{-\frac{1}{4}\left(7-4y^2\right)^{3/2} + C}$$

(e) (10 pts) $\int x^3\sqrt{x^2+1}\,dx$

$u = x^2+1 \Rightarrow x^2 = u-1$

$du = 2x\,dx$

$$= \int x^3\sqrt{u}\left(\frac{du}{2x}\right) = \frac{1}{2}\int x^2\sqrt{u}\,du = \frac{1}{2}\int (u-1)\sqrt{u}\,du$$

$$= \frac{1}{2}\int u^{3/2} - u^{1/2}\,du = \frac{1}{2}\left[\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right] + C$$

$$= \boxed{\frac{1}{5}(1+x^2)^{5/2} - \frac{1}{3}(1+x^2)^{3/2} + C}$$

7. (10 pts) Use the Max-Min Inequality to find upper and lower bounds for the value of

$$\int_0^1 \frac{1}{1+x^3}\,dx$$

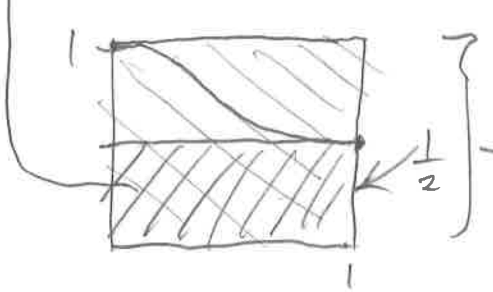
Since $(\min f)(b-a) < \int_a^b f(x)\,dx < (\max f)(b-a)$

We have $a=0, b=1$, and $\min f = \frac{1}{2}$ and $\max f = 1$

$$\frac{1}{2}(1-0) < \int_0^1 \frac{1}{1+x^3}\,dx < 1\cdot(1-0)$$

$$\frac{1}{2} < \int_0^1 \frac{1}{1+x^3}\,dx < 1$$

Note:



8. Given the formulas for the sums of powers of integers, answer the following questions.

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2} \text{ and } \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$$

(a) (10 pts) Given the function $f(x) = x^2$, find a formula for the lower sum obtained by dividing the interval $[0, b]$ into n equal subintervals.

$$h = \frac{b-0}{n} = \frac{b}{n}$$

$$x_k = 0 + kh = \frac{kb}{n}$$

Lower sum

$$L_n = \sum_{k=0}^{n-1} f(x_k)\, h = \sum_{k=0}^{n-1} \left(\frac{kb}{n}\right)^2 \frac{b}{n} = \frac{b^3}{n^3} \sum_{k=0}^{n-1} k^2$$

$$= \frac{b^3}{n^3} \sum_{k=1}^{n-1} k^2 = \frac{b^3}{n^3}\left(\frac{(n-1)(n-1+1)(2(n-1)+1)}{6}\right)$$

$$= \frac{b^3}{n^3} \frac{n(n-1)(2n-1)}{6}$$

$$= \frac{1}{6} b^3 \frac{(n-1)(2n-1)}{n^2} = \boxed{\frac{b^3}{6}\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)}$$

(b) (10 pts) Take the limit of the sums in part 8a as $n \to \infty$ to calculate the area under the curve over $[0, b]$.

$$L = \lim_{n \to \infty} L_n = \lim_{n \to \infty} \frac{b^3}{6}\left(1 - \frac{1}{n}\right)\left(2 - \frac{1}{n}\right)$$

$$= \frac{b^3}{6}(1 - 0)(2 - 0)$$

$$= \boxed{\frac{b^3}{3}}$$

9. (10 pts) Find the derivative with respect to x of the integral

$$\frac{d}{dx}\int_1^{\sin x} 3t^2\, dt$$

Combining Fundamental thm of Calulus with the chain rule gives.

$$f(x) = \int_a^x g(v)\,dv \Rightarrow f'(x) = g(x)$$

with the chain Rule it is

$$f(x) = \int_a^{u(x)} g(v)\,dv \Rightarrow f'(x) = g(u(x))\,u'(x)$$

So $g(v) = 3v^2$ and $u(x) = \sin x$

So $$\frac{d}{dx}\int_1^{\sin x} 3t^2\,dt = 3(\sin x)^2 \cos x$$

$$= \boxed{3\sin^2 x \cos x}$$

2nd way Integrate, then differentiate

$$\int_1^{\sin x} 3t^2\,dt = \left.\frac{3t^3}{3}\right]_1^{\sin x} = \sin^3 x$$

So $$\frac{d}{dx}\sin^3 x = 3\sin^2 x\,(\cos x)$$

$$= \boxed{3\sin^2 x \cos x}$$