Math 113: Trigonometric Identities

Chapter 6, 7, 8, 9

Many of the trigonometric identities can be derived in succession from the identities:

$$\sin(-x) = -\sin x, \tag{1}$$

$$\cos(-x) = \cos x, \tag{2}$$

$$\sin(x+y) = \sin x \cos y + \sin y \cos x, \text{ and} \tag{3}$$

$$\cos(x+y) = \cos x \cos y - \sin x \sin y. \tag{4}$$

The first and second identities indicate that sin and cos are odd and even functions, respectively. Suppose that $y = -w$, then (3) simplifies to

$$\begin{aligned} \sin(x + (-w)) &= \sin x \cos(-w) + \sin(-w) \cos x && \text{by (3)} \\ &= \sin x \cos w - \sin w \cos x && \text{by (1) and (2)} \end{aligned}$$

Since w is an arbitrary label, then y will do as well. Hence,

$$\sin(x-y) = \sin x \cos y - \sin y \cos x \tag{5}$$

Similarly, equation (4) simplifies as

$$\cos(x-y) = \cos x \cos y + \sin x \sin y \tag{6}$$

The Double Angle identities can be derived from equations (3) and (4). Suppose $x = y$, then (3) simplifies as

$$\sin(x+x) = \sin x \cos x + \sin x \cos x.$$

Hence,

$$\sin(2x) = 2 \sin x \cos x. \tag{7}$$

Similarly,

$$\cos(2x) = \cos^2 x - \sin^2 x. \tag{8}$$

The first of the Pythagorean identities can be found by setting $x = y$ in (6). Hence,

$$\cos(x-x) = \sin x \sin x + \cos x \cos x.$$

So,

$$\sin^2 x + \cos^2 x = 1. \tag{9}$$

Dividing both sides of (9) by $\cos^2 x$ yields

$$\tan^2 x + 1 = \sec^2 x. \tag{10}$$

Dividing both sides of (9) by $\sin^2 x$ yields

$$1 + \cot^2 x = \csc^2 x. \tag{11}$$

Equations (8) and (9) can generate the Power Reductions formulas. Using $\cos^2 x = 1 - \sin^2 x$, (8) can be written as

$$\cos(2x) = (1 - \sin^2 x) - \sin^2 x = 1 - 2\sin^2 x.$$

Solving the above equation for $\sin^2 x$ yields

$$\sin^2 x = \frac{1 - \cos(2x)}{2}. \tag{12}$$

Similarly,

$$\cos^2 x = \frac{1 + \cos(2x)}{2}. \tag{13}$$

The product identities can be found using equations (3) through (6). For example, adding (3) and (5) yields

$$\sin(x - y) + \sin(x + y) = \sin x \cos y + \sin y \cos x + \sin x \cos y - \sin x \cos y$$

$$\sin(x - y) + \sin(x + y) = 2 \sin x \cos y.$$

Hence,

$$\sin x \cos y = \frac{1}{2} \left[\sin(x - y) + \sin(x + y) \right]. \tag{14}$$

Similarly,

$$\cos x \cos y = \frac{1}{2} \left[\cos(x - y) + \cos(x + y) \right] \text{ and} \tag{15}$$

$$\sin x \sin y = \frac{1}{2} \left[\cos(x - y) - \cos(x + y) \right]. \tag{16}$$