

1.2

Conditionals and Biconditionals

DEF: For prop. P and Q , the conditional sentence

$P \Rightarrow Q$ is the prop. "If P , then Q "

Prop P is called the antecedent,

Prop Q is called the consequent.

The cond. sentence $P \Rightarrow Q$ is true

if and only if P is false or Q is true.

Truth table:

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

Promise!

When antecedent is false,
then the promise is
always true.

"If $1 + 1 = 2$, then I will give you a dollar"

Note that the sentence can be true even if
there is no connection between P & Q .

For example:

Mars has 10 moons $\Rightarrow 1 + 1 = 2$ is true!

Whereas

$1 + 2 = 3 \Rightarrow 1 < 0$ is false.

DEF: Let P and Q be propositions.

The converse of $P \Rightarrow Q$ is $Q \Rightarrow P$

The contrapositive of $P \Rightarrow Q$ is $(\sim Q) \Rightarrow (\sim P)$

What is the converse & contrapositive of

If f is differentiable at x_0 , then
 f is continuous at x_0 .

Q

So converse is $Q \Rightarrow P$

If f is continuous at x_0 , then f is diff. at x_0 .
and contrapositive is $\sim Q \Rightarrow \sim P$

If f is not cont. at x_0 , then f is not diff. at x_0 .

Thm 1.1 (a) a conditional sentence and its contrapositive are equivalent
(b) a conditional sentence and its converse are not equivalent.

Proof:

P	Q	$\sim P$	$\sim Q$	$P \Rightarrow Q$	$Q \Rightarrow P$	$\sim Q \Rightarrow \sim P$
T	T	F	F	T	T	T
T	F	F	T	F	F	F
F	T	T	F	T	T	T
F	F	T	T	T	T	T

DEF: For prop. P and Q the biconditional sentence $P \Leftrightarrow Q$ is the prop. " P if and only if Q ."

$P \Leftrightarrow Q$ is true exactly when P and Q have the same truth values.

Truth table

P	Q	$P \Leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

The propositional forms P and Q are equivalent precisely when $P \Leftrightarrow Q$ is a tautology.

One key to success in mathematics is the ability to replace a statement by a more useful or enlightening one. For example:

$$x^2 - 7x = -12 \Leftrightarrow x^2 - 7x + 12 = 0$$

$$\Leftrightarrow (x-4)(x-3) = 0$$

$$\Leftrightarrow x-4=0 \text{ or } x-3=0$$

$$\Leftrightarrow x=4 \text{ or } x=3$$

Thm:

For Prop: P, Q, R

- (a) $P \Rightarrow Q$ is equivalent to $(\neg P) \vee Q$
(b) $P \Leftrightarrow Q$ is equivalent to $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$
(c) $\neg(P \wedge Q)$ is " " " $(\neg P) \vee (\neg Q)$
(d) $\neg(P \vee Q)$ " " " $(\neg P) \wedge (\neg Q)$
(e) $\neg(P \Rightarrow Q)$ " " " $P \wedge (\neg Q)$
(f) $\neg(P \wedge Q)$ " " " $P \Rightarrow \neg Q$
(g) $P \wedge (Q \vee R)$ " " " $(P \wedge Q) \vee (P \wedge R)$
(h) $P \vee (Q \wedge R)$ " " " $(P \vee Q) \wedge (P \vee R)$

(Proof: Exercise 7)

Connectives in English

Use $P \Rightarrow Q$ to translate:

If P , then Q

P implies Q

P is sufficient for Q

P only if Q

Q , if P

Q whenever P

Q is necessary for P

Q , when P

Use $P \Leftrightarrow Q$ to translate:

P if and only if Q

P if, but only if, Q

P is equiv. to Q

P is necessary and suff for Q .

Conventions: connectives

$\neg \wedge \vee \Rightarrow \Leftrightarrow$ are applied in order listed.

Ex: $P \vee \neg Q \Leftrightarrow R \Rightarrow S$ same as $[P \vee (\neg Q)] \Leftrightarrow (R \Rightarrow S)$