

Section 1.3

Quantifiers

$x \geq 3 \rightarrow$ not a proposition unless x is known.

$\rightarrow$ called an "open sentence" or predicate

Notation:

P is an open sentence

variables $x_1, x_2, \dots, x_k$

$P(x_1, x_2, \dots, x_k) =$ some open sentence

Ex:

$P(x, y, z) = "x + y = z"$

Then $P(1, 2, 4)$ is false

$P(2, 3, 5)$ is true

The collection of objects that may be substituted to make an open sentence a true proposition is called the truth set of the sentence.

universe of discourse - the entire collection of objects to be used to determine the truth set.

Number systems:

$\mathbb{N}$ = natural numbers

$\mathbb{Z}$ = integers

$\mathbb{Q}$ = rationals

$\mathbb{R}$ = reals

DEF: With a universe specified, two open sentences $P(x)$ and $Q(x)$ are equivalent iff they have the same truth set.

EX: " $3x + 2 = 20$ " is equiv to " $x = 6$ "
no matter which number system is
used as the universe.

EX: " $x^2 = 9$ " is not equivalent to " $x = 3$ "
when the universe is the integers.

Note that it is when the universe is $\mathbb{N}$.

DEF: For an open sentence $P(x)$, the sentence $(\exists x) P(x)$ is read "there exists x such that $P(x)$ " or "for some x , $P(x)$ " and is true iff the truth set of $P(x)$ is nonempty. The symbol $\exists$ is called the existential quantifier.

EX: $(\exists x)(x \geq 3)$

$(\exists x)(|x| > 0)$

DEF: For an open sentence $P(x)$, the sentence $(\forall x) P(x)$ is read "for all x , $P(x)$ " and is true iff the truth set of $P(x)$ is the entire universe. The symbol $\forall$ is called the universal quantifier.

Ex: $(\forall x)(x > 0 \vee x = 0 \vee x < 0)$

$$(\forall x)(x < x - 1)$$

$$(\forall x)(\sin^2 x \geq 0)$$

Sentences of the form

$$\text{"All } P(x) \text{ are } Q(x)\text{"} \Leftrightarrow (\forall x)(P(x) \Rightarrow Q(x))$$

$$\text{"Some } P(x) \text{ are } Q(x)\text{"} \Leftrightarrow (\exists x)(P(x) \wedge Q(x))$$

Ex: "For every odd prime x less than 10,

$x^2 + 4$ is prime" write symbolically:

$$(\forall x)[x \text{ is prime} \wedge x \text{ is odd} \wedge x < 10 \Rightarrow x^2 + 4 \text{ is prime}]$$

DEF: Two quantified sentences are equivalent in a given universe iff they have the same truth value in that universe. Two quantified sentences are equivalent iff they are equivalent in every universe.

Thm 1.3

If $A(x)$ is an open sentence with var x , then

(a) $\sim(\forall x)A(x)$ is equivalent to $(\exists x) \sim A(x)$

(b) $\sim(\exists x)A(x)$ is equivalent to $(\forall x) \sim A(x)$

Useful for finding denials.

For example: "All primes are odd"

symbolized as $(\forall x)(x \text{ is prime} \Rightarrow x \text{ is odd})$

The negation is $\sim(\forall x)(x \text{ is prime} \Rightarrow x \text{ is odd})$

$$\Leftrightarrow (\exists x)(\sim(x \text{ is prime} \Rightarrow x \text{ is odd}))$$

$$\Leftrightarrow (\exists x)(\sim(x \text{ is prime}) \vee x \text{ is odd})$$

$$\Leftrightarrow (\exists x)(x \text{ is prime} \wedge \sim(x \text{ is odd}))$$

DEF:

For an open sentence $P(x)$, the proposition $(\exists! x) P(x)$ is read "there exists a unique x such that $P(x)$ " and is true iff the truth set of $P(x)$ has exactly one element. The symbol $\exists!$ is called the uniqueness existence quantifier.

Ex: $(\exists! x) (x \text{ is prime} \wedge x \text{ is even})$

Thm: If $A(x)$ is an open sentence with var x , then

(a) $(\exists! x) A(x) \Rightarrow (\exists x) A(x)$

(b) $(\exists! x) A(x)$ is equivalent to

$$(\exists x) A(x) \wedge (\forall y)(\forall z) (A(y) \wedge A(z) \Rightarrow y=z)$$