

1.5 Basic Proof Methods II

indirect proofs

first: Proof by Contraposition

Uses the tautology $(P \Rightarrow Q) \Leftrightarrow (\sim Q \Rightarrow \sim P)$

Proof by Contraposition of $P \Rightarrow Q$

Proof:

Assume $\sim Q$

∴
Therefore, $\sim P$

Thus, $\sim Q \Rightarrow \sim P$

Therefore $P \Rightarrow Q$

Ex: Let m be an integer. Prove that
if m^2 is odd, then m is odd.

Proof:

Suppose m is even. Then $m = 2k$ for some integer k . It follows that $m^2 = 4k^2 = 2(2k^2)$.
Therefore, m^2 is even. Thus, if m is even, then m^2 is even. Therefore, if m^2 is odd, then m is odd.

Proof by contradiction

Uses the tautology $P \Leftrightarrow (\neg P \Rightarrow (Q \wedge \neg Q))$

To prove P , we first assume P is false, then see what happens. If what happens is impossible, then we have a contradiction. It follows that P must be true.

Ex: Prove that the graphs of $y = x^2 + x + 2$ and $y = x - 2$ do not intersect.

Proof: Suppose that the graphs of $y = x^2 + x + 2$ and $y = x - 2$ intersect at some point (a, b) . It follows that (a, b) lies on both graphs so

$$b = a^2 + a + 2 \quad \text{and} \quad b = a - 2$$

$$\Rightarrow a^2 + a + 2 = a - 2$$

$$\Rightarrow a^2 = -4$$

But this is impossible, since the square of a ~~real~~ real number cannot be negative. It follows that the graphs of $y = x - 2$ and $y = x^2 + x + 2$ do not intersect.

Proof By Contradiction
Suppose $\neg P$

Therefore, Q .

Therefore, $\neg Q$

Hence $Q \wedge \neg Q$, a contradiction

Thus, P

Note that proof by contradiction may require a "spark of insight" to determine a useful Q.

Ex: $\sqrt{2}$ is an irrational number.

Proof: Suppose $\sqrt{2}$ is a rational number. Then it follows that $\sqrt{2} = \frac{s}{t}$, where s & t are integers.

Thus $2 = \frac{s^2}{t^2}$. Since s^2 & t^2 are squares, then s^2 contains an even number of 2's as prime factors (This is Q) and t^2 contains an even number of 2's. Since $s^2 = 2t^2$, then it follows that s^2 contains an odd number of 2's ($\sim$ Q). This is a contradiction, since s^2 cannot have both an even and an odd number of 2's as factors. we conclude that $\sqrt{2}$ is irrational.

TWO PART PROOF of $P \Leftrightarrow Q$

(i) Show $P \Rightarrow Q$ by any method

(ii) Show $Q \Rightarrow P$ by any method

Therefore, $P \Leftrightarrow Q$

Ex: Let a be a prime, and b and c be positive integers. Prove that a divides the product bc if and only if a divides b or a divides c .

Proof:

Show $P \Rightarrow Q$

Suppose a is prime and a divides bc . By the Fund. thm of arithmetic, b & c can be written uniquely as a product of primes

Then $b = p_1 \cdot p_2 \cdot \dots \cdot p_k$
 $c = q_1 \cdot q_2 \cdot \dots \cdot q_r$

and $bc = p_1 \cdot p_2 \cdot \dots \cdot p_k \cdot q_1 \cdot q_2 \cdot \dots \cdot q_r$

Since a is prime and divides bc , then a is one of the prime factors of bc . Thus,

either $a = p_i$ for some i or $a = q_j$ for some j .

If $a = p_i$, then a divides b . If $a = q_j$, then a divides c . Therefore, a divides b or a divides c .

(ii) show $Q \Rightarrow P$.

Suppose a divides b or a divides c . If a divides b , then a divides bc . If a divides c , then a divides bc .

Some statements are undecidable. You can't make a decision based on the axioms you have.

Ex: The fifth post. of Euclid's geometry.

"Given a line and a point not on the line, exactly one line can be drawn through the point parallel to the line".

This post. cannot be proved using the other axioms. It is undecidable. There are theories of geometry where this postulate are assumed false. (Riemannian & Lobachevskian Geometry).