

1.6 Proofs involving Quantifiers

Note: Most theorems involve quantifiers, even though the quantifier may not appear explicitly in the Statement.

For example: If a divides b , and b divides c , then a divides c . It involves quantifiers since

$$(\forall a)(\forall b)(\forall c) [(a \text{ divides } b) \wedge (b \text{ divides } c) \Rightarrow (a \text{ divides } c)]$$

Existence theorems

Construction proofs name them:

Ex: Prove "There is an even prime number"

Proof: $(\exists x)(x \text{ is prime and } x \text{ is even})$.

Observe that 2 is even and is prime

Ex: Prove that there exists a natural number whose 4th power is the sum of 3 other 4th powers.

Proof: 20615673!

$$20615673^4 = 2682440^4 + 15365639^4 + 18796760^4$$

Proof of $(\exists x) P(x)$ by CONTRADICTION

Proof:

Suppose $\sim(\exists x) P(x)$.

Then $(\forall x) \sim P(x)$

$\vdots$

Therefore, $\sim Q \wedge Q$, a contradiction

Thus, $\sim(\exists x) P(x)$ is false.

Therefore, $(\exists x) P(x)$ is true

sometimes you prove the existence without ever producing the particular object.

Ex: Rolle's Thm, Mean Value Theorem.

Ex: Prove the polynomial

$$r(x) = x^{69} - x^5 + 3$$

has at least one real root.

Proof: we must show that $(\exists t) (t \text{ is real} \ \& \ r(t)=0)$

By FT of Algebra, $r(x)$ has 71 zeros that are real or complex. Since the poly has real coef, its nonreal zeros come in pairs.

Hence there is an even number of complex solutions, which leaves an odd number of real roots.

Hence, $r(x)$ has at least one real root.

Direct Proof of $(\forall x)P(x)$

Proof:

Let x be an arbitrary object in the universe. (The universe should be named or its objects described.)

⋮

Hence, $P(x)$ is true.

Since x is arbitrary, $(\forall x)P(x)$ is true.

Ex: If x is an even integer, then x^2 is an even integer.

Proof: (Note: the universe is all integers)

Let x be an integer. (An arbitrary member of the universe)

Suppose x is even. Then $x = 2k$.

Therefore $x^2 = 2(2k^2)$. Since $2k^2$ is an integer, then x^2 is even.

Note: This is not the same as:

Proof: Let x be an arbitrary integer, say 10.

If x is even (10 is even), then x^2 is

even ($10^2 = 100$ is even). Since 10 is

arbitrary then if x is even, then x^2 is even

Proof of $(\forall x) P(x)$ by Contradiction

Suppose $\neg(\forall x) P(x)$

Then $(\exists x) \neg P(x)$

Let t be an object such that $\neg P(t)$

$\vdots$

Therefore $Q \wedge \neg Q$

Thus $(\exists x) \neg P(x)$ is false; so its denial $(\forall x) P(x)$ is true

Ex: If $0 < x < \pi/2$, $\sin x + \cos x > 1$

Suppose that there exists an $x \in (0, \pi/2)$ such that

$$\sin x + \cos x \leq 1$$

$$\text{Then } (\sin x + \cos x)^2 \leq 1 \quad (\sin x > 0 \text{ and } \cos x > 0)$$

$$\text{But then } \sin^2 x + 2\cos x \sin x + \cos^2 x \leq 1$$

This is equivalent to $2\cos x \sin x \leq 0$

But we have a contradiction since $\cos x > 0$ and $\sin x > 0$ for $x \in (0, \pi/2)$. It must be that if

$$x \in (0, \pi/2), \quad \sin x + \cos x > 1.$$

When more than one quantifiers are used, great care must be taken. They may not be valid!

These are good.

$$1. (\forall x)(\forall y) P(x, y) \Leftrightarrow (\forall y)(\forall x) P(x, y)$$

$$2. (\exists x)(\exists y) P(x, y) \Leftrightarrow (\exists y)(\exists x) P(x, y)$$

$$3. [(\forall x)P(x) \vee (\forall x)Q(x)] \Rightarrow (\forall x)(P(x) \vee Q(x))$$

$$4. (\forall x)[P(x) \Rightarrow Q(x)] \Rightarrow [(\forall x)P(x) \Rightarrow (\forall x)Q(x)]$$

$$5. (\forall x)[P(x) \wedge Q(x)] \Leftrightarrow [(\forall x)P(x) \wedge (\forall x)Q(x)]$$

$$6. (\exists x)(\forall y)(P(x, y)) \Rightarrow (\forall y)(\exists x) P(x, y)$$

Not Valid:

$$1. (\exists x) P(x) \Rightarrow (\forall x) P(x) \quad (\text{proof by example})$$

$$2. (\forall x)[P(x) \vee Q(x)] \Rightarrow [(\forall x)P(x) \vee (\forall x)Q(x)]$$

$$3. [(\forall x)P(x) \Rightarrow (\forall x)Q(x)] \Rightarrow (\forall x)[P(x) \Rightarrow Q(x)]$$

$$4. (\forall y)(\exists x) P(x, y) \Rightarrow (\exists x)(\forall y) P(x, y)$$

Proof of $(\exists! x) P(x)$

Proof:

- (i) Prove that $(\exists x) P(x)$ by any method.
(ii) Prove that $(\forall y)(\forall z)[P(y) \wedge P(z) \Rightarrow y=z]$

assume that $y \neq z$ are objects in the universe
such that $P(y)$ and $P(z)$ are true.

⋮

thus $y=z$

From (i) & (ii) conclude $(\exists! x) P(x)$ is true.