

Basic Concepts of Set Theory

2.1

- A set is a specified collection of objects.
- The objects in a set are called elements (or members).
- For any object, there must be a definite yes or no answer to the question of whether the object is a member of the set.
- Capital letters denote sets.
- lowercase letters denote elements or members.

$x \in A \rightarrow x$ is an element of the set A

$x \notin A \rightarrow x$ is not an element of the set A

You can list all the elements in a set or describe in words what belongs to it.

Ex:

$P(x) = "x \text{ is an odd integer between } 0 \text{ and } 12"$

$$A = \{x : P(x)\} = \{1, 3, 5, 7, 9, 11\}$$

Reminders:

$$\mathbb{N} = \{1, 2, 3, \dots\} \quad (\text{Natural numbers})$$

$$\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\} \quad (\text{integers})$$

$$\mathbb{Q} = \{x : a, b \in \mathbb{Z} \quad x = \frac{a}{b}\} \quad (\text{Rational numbers})$$

$$\mathbb{R} = \text{Real numbers}$$

Notation for Intervals

$[a, b] = \{x: x \in \mathbb{R} \text{ and } a \leq x \leq b\}$ (closed interval from a to b)

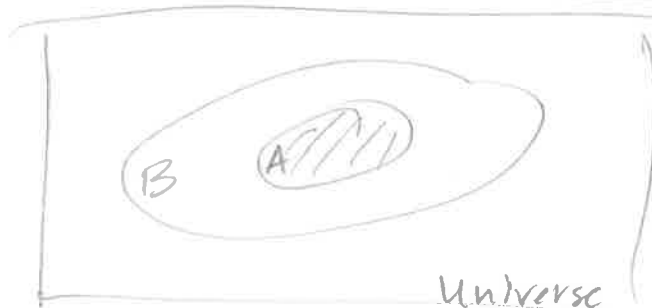
$(a, b) = \{x: x \in \mathbb{R} \text{ and } a < x < b\}$ (open interval from a to b)

$(a, \infty) = \{x: x \in \mathbb{R} \text{ and } x > a\}$
 $(-\infty, a) = \{x: x \in \mathbb{R} \text{ and } x < a\}$ } open rays.

$[a, \infty) = \{x: x \in \mathbb{R} \text{ and } x \geq a\}$
 $(-\infty, a] = \{x: x \in \mathbb{R} \text{ and } x \leq a\}$ } closed rays.

DEF: Let $\emptyset = \{x: x \neq x\}$, Then $\emptyset$ is a set with no elements and is called the empty set

DEF: Let A and B be sets. We say that A is a subset of B iff every element of A is also an element of B .
In symbols, $A \subseteq B \Leftrightarrow (\forall x)(x \in A \Rightarrow x \in B)$



Direct Proof of $A \subseteq B$

Let x be any object

Suppose $x \in A$

!

Thus $x \in B$

Therefore $A \subseteq B$

Thm: (2.1) For any set A ,

(a) $\emptyset \subseteq A$

(b) $A \subseteq A$

Proper subset: $A \subseteq B$, but $A \neq B$.

Denoted $A \subset B$

Thm 2.2 Let A, B, C be sets.

If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

DEF:

Let A and B be sets, Then $A = B$ iff
 $A \subseteq B$ and $B \subseteq A$.

Ex: Show $A = \{x: x \text{ is a solution to } x^2 - 1 = 0\}$ and $A = B$ $B = \{-1, 1\}$

(i) show $B \subseteq A$

By substituting -1 & 1 into $x^2 - 1 = 0$, we find that -1 & 1 are both solutions to $x^2 - 1 = 0$. Hence, $B \subseteq A$.

(ii) show $A \subseteq B$. Let $t \in A$. By def of T , t is a solution to $x^2 - 1 = 0$. Thus, $t^2 - 1 = 0$. Factoring leads to $(t-1)(t+1) = 0$. Since the product is zero, then $t-1=0$ or $t+1=0$. Thus $t = -1$ or $t = 1$. It follows that $t \in B$. Therefore, $A \subseteq B$.

By (i) and (ii), it follows that $A = B$.

Thm: If A & B are sets with no elements, then $A = B$.

Proof: Since A has no elements, then the sentence $(\forall x)(x \in A \Rightarrow x \in B)$ is true. Similarly, $(\forall x)(x \in B \Rightarrow x \in A)$ is true. Thus $A \subseteq B$ and $B \subseteq A$. We conclude that $A = B$.

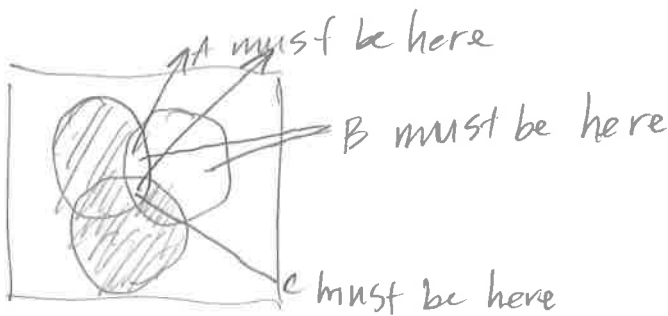
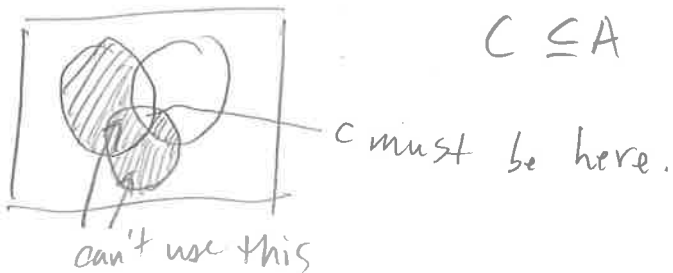
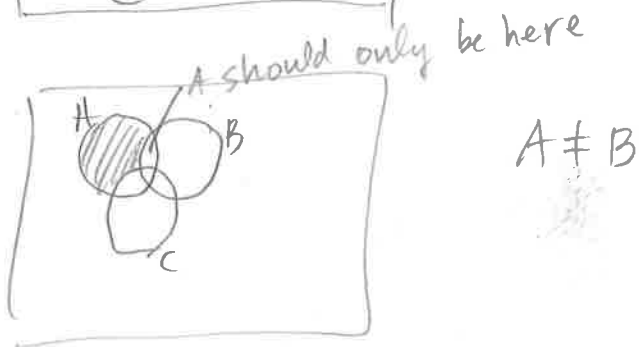
Using Venn Diagrams

Helpful for working out details in proofs (But they aren't proofs!)

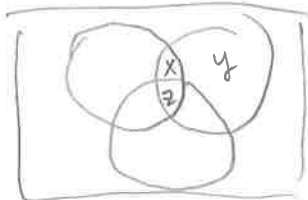
Ex: Construct nonempty sets A, B, C such that

$$\begin{aligned} A &\subseteq B \\ A &\neq B \\ C &\subseteq A \\ \text{and } A &\not\subseteq C \end{aligned}$$

Satisfies H!



So



$$\begin{aligned} A &= \{x, z\} \\ B &= \{x, y, z\} \\ C &= \{z\} \end{aligned}$$

DEF! Let A be a set. The power set of A is the set whose elements are the subsets of A and is denoted $\mathcal{P}(A)$. Thus,

$$\mathcal{P}(A) = \{B : B \subseteq A\}$$

Ex: Let $A = \{a, b, c, d\}$

Then

$$\begin{aligned}\mathcal{P}(A) = \{ & \emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \\ & \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \\ & \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, \\ & \{a, b, c, d\} \}\end{aligned}$$

Thm: If A is a set with n elements, then $\mathcal{P}(A)$ is a set with 2^n elements.

Thm: Let A and B be sets. Then $\underbrace{A \subseteq B}_P$ iff $\underbrace{\mathcal{P}(A) \subseteq \mathcal{P}(B)}_Q$.

Proof: Q

(i) show $P \Rightarrow Q$

Suppose $A \subseteq B$ and $X \in \mathcal{P}(A)$. Since $X \in \mathcal{P}(A)$, then $X \subseteq A$. Because $A \subseteq B$, then $X \subseteq B$, ^{Thm 2.2} which means that $X \in \mathcal{P}(B)$. Thus $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.

(ii) Assume $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.

By Thm 2.1, $A \subseteq A$, which means $A \in \mathcal{P}(A)$.

Since $\mathcal{P}(A) \subseteq \mathcal{P}(B)$, then $A \in \mathcal{P}(B)$. Therefore,
 $A \subseteq B$.

Bertrand Russell: Russel Paradox.

1902 letter British mathematician & philosopher

will see again in chap 5.