

2.3 Extended Set Operations and Indexed Families of Sets.

Script letters denote families of sets.

DEF: Let $\mathcal{A}$ be a family of sets. The union over $\mathcal{A}$ is

$$\bigcup_{A \in \mathcal{A}} A = \{x : x \in A \text{ for some set } A \in \mathcal{A}\}$$

DEF: Let $\mathcal{A}$ be a family of sets. The intersection over $\mathcal{A}$ is

$$\bigcap_{A \in \mathcal{A}} A = \{x : x \in A \text{ for every } A \in \mathcal{A}\}$$

Note: When is an object in these sets?

$$x \in \bigcup_{A \in \mathcal{A}} A \quad \text{iff} \quad (\exists A)(A \in \mathcal{A} \wedge x \in A)$$

$$x \in \bigcap_{A \in \mathcal{A}} A \quad \text{iff} \quad (\forall A)(A \in \mathcal{A} \Rightarrow x \in A)$$

EX: $\mathcal{A} = \{\{r, k, s, t, a\}, \{k, d, s\}\}$

$$\bigcup_{A \in \mathcal{A}} A = \{r, k, s, t, a, d\}$$

$$\bigcap_{A \in \mathcal{A}} A = \{k, s\}$$

Ex: $\mathcal{A} = \{A : A = (-a, a) \wedge a \in \mathbb{R}\}$

$$\bigcap_{A \in \mathcal{A}} A = \{0\}$$

$$\bigcup_{A \in \mathcal{A}} A = \mathbb{R}$$

Thm For every set B in a family $\mathcal{A}$ of sets,

(a) $\bigcap_{A \in \mathcal{A}} A \subseteq B$

(b) $B \subseteq \bigcup_{A \in \mathcal{A}} A$

DEF Let Δ be a nonempty set such that for each $\alpha \in \Delta$ there is a corresponding set A_α . The family $\{A_\alpha : \alpha \in \Delta\}$ is an indexed family of sets. The set Δ is called the indexing set and each $\alpha \in \Delta$ is an index.

Ex: Let $\Delta = \mathbb{N}$

$(\forall n \in \mathbb{N})$ Define $A_n = \{n, n+1, 2n\}$

$$A_1 = \{1, 2\}$$

$$A_2 = \{2, 3, 4\}$$

$$A_3 = \{3, 4, 6\}$$

$$A_{10} = \{10, 11, 20\}$$

$$A_{40} = \{40, 41, 80\}$$

Ex 1

$$A_1 = \{1, 2, 4, 5\}$$

$$A_2 = \{2, 3, 5, 6\}$$

$$A_3 = \{3, 4, 5, 6\}$$

$$\Delta = \{1, 2, 3\}$$

$$\begin{aligned} \mathcal{A} &= \{A_1, A_2, A_3\} \\ &= \{A_i : i \in \Delta\} \end{aligned} \quad \xleftrightarrow{\text{same}}$$

$$A_{10} = \{1, 2, 4, 5\}$$

$$A_{22} = \{2, 3, 5, 6\}$$

$$A_{\pi} = \{3, 4, 5, 6\}$$

$$\Gamma = \{10, 22, \pi\}$$

$$\begin{aligned} \mathcal{A} &= \{A_{10}, A_{22}, A_{\pi}\} \\ &= \{A_i : i \in \Gamma\} \end{aligned}$$

Ex 1

$$\Delta = \{0, 1, 2, 3, 4\}$$

$$(\forall x \in \Delta) \quad A_x = \{2x + 4, 8, 12 - 2x\}$$

$$\Rightarrow A_0 = \{4, 8, 12\}$$

$$A_1 = \{6, 8, 10\}$$

$$A_2 = \{8\}$$

$$A_3 = \{6, 8, 10\}$$

$$A_4 = \{4, 8, 12\}$$

$$\Delta = \{0, 1, 2, 3, 4\} \rightarrow 5 \text{ elements.}$$

But $\mathcal{A} = \{A_i : i \in \Delta\}$ only contains 3 elements

Note the difference in notations:

$$\bigcup_{\alpha \in \Delta} A_\alpha = \bigcup_{A \in \mathcal{A}} A \quad \text{and}$$

$$x \in \bigcup_{\alpha \in \Delta} A_\alpha \text{ iff } (\exists \alpha)(\alpha \in \Delta \wedge x \in A_\alpha)$$

$$\bigcap_{\alpha \in \Delta} A_\alpha = \bigcap_{A \in \mathcal{A}} A \quad \text{and}$$

$$x \in \bigcap_{\alpha \in \Delta} A_\alpha \text{ iff } (\forall \alpha)(\alpha \in \Delta \Rightarrow x \in A_\alpha)$$

Previously, $\longrightarrow$

$$x \in \bigcup_{A \in \mathcal{A}} A \text{ iff } (\exists A)(A \in \mathcal{A} \wedge x \in A)$$

$$x \in \bigcap_{A \in \mathcal{A}} A \text{ iff } (\forall A)(A \in \mathcal{A} \Rightarrow x \in A)$$

Thm 2.9

$$(a) \bigcap_{\alpha \in \Delta} A_\alpha \subseteq A_\beta \text{ for each } \beta \in \Delta$$

$$(b) A_\beta \subseteq \bigcup_{\alpha \in \Delta} A_\alpha \text{ for each } \beta \in \Delta.$$

$$(c) \widetilde{\bigcap_{\alpha \in \Delta} A_\alpha} = \bigcup_{\alpha \in \Delta} \widetilde{A}_\alpha \quad \left. \vphantom{\bigcup_{\alpha \in \Delta} \widetilde{A}_\alpha} \right\} \text{De Morgan's Laws.}$$

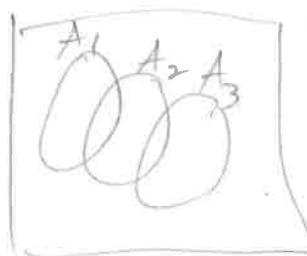
$$(d) \widetilde{\bigcup_{\alpha \in \Delta} A_\alpha} = \bigcap_{\alpha \in \Delta} \widetilde{A}_\alpha$$

DEF The indexed family $\mathcal{A} = \{A_\alpha : \alpha \in \Delta\}$ of sets is pairwise disjoint iff $(\forall \alpha, \beta \in \Delta) (\text{If } A_\alpha \neq A_\beta, \text{ then } A_\alpha \cap A_\beta = \emptyset)$.

Ex:



Pairwise disjoint



Not pairwise disjoint
but $A_1 \cap A_2 \cap A_3 = \emptyset$