

2.4 Induction

Properties of the Natural numbers

1. Successor Properties

1 is a natural number

Every natural number x has a unique successor $x+1$

1 is not the successor of any natural number.

2. Closure Properties

For each pair of natural numbers, the sum $x+y$ is a natural number

For each pair of natural numbers, the product xy is a natural number.

3. Associative prop.

$$(\forall x, y, z \in \mathbb{N}) (x + (y + z) = (x + y) + z)$$

$$(\forall x, y, z \in \mathbb{N}) (x(yz) = (xy)z)$$

4. Commutativity properties

$$(\forall x, y \in \mathbb{N}) [x + y = y + x \text{ and } xy = yx]$$

5. Dist. Properties

$$(\forall x, y, z \in \mathbb{N}) (x(y+z) = xy + xz)$$

$$(\forall x, y, z \in \mathbb{N}) ((y+z)x = yx + zx)$$

6. Cancellation prop.

$$(\forall x, y, z \in \mathbb{N}) \left[\begin{array}{l} x+z = y+z \Rightarrow x=y \\ xz = yz \Rightarrow x=y \end{array} \right]$$

7. Order Properties.

$$(\forall x, y, z \in \mathbb{N}) \left[\begin{array}{l} x < y \text{ iff } (\exists w \in \mathbb{N}) (x + w = y) \\ x \leq y \text{ iff } x < y \text{ or } x = y \\ x < y \text{ and } y < z \Rightarrow x < z \\ x \leq y \text{ and } y \leq x \Rightarrow x = y \end{array} \right]$$

$$x < y \Rightarrow \begin{array}{l} x+z < y+z \\ xz < yz \end{array}$$

Principle of Mathematical Induction (PMI)

If S is a subset of N with these two properties:

- (i) $1 \in S$
- (ii) $(\forall n \in N) (n \in S \Rightarrow n+1 \in S)$

then $S = N$.

A set S of natural numbers is called an inductive set iff it has the property that whenever $n \in S$, then $n+1 \in S$.

inductive definitions

Ex: factorials:

- (i) define $1! = 1$
- (ii) $(\forall n \in N) ((n+1)! = (n+1)n!)$ (recursive def.)

Proof using the principle of Mathematical Induction

Let $S = \{n \in N : \text{the statement is true for } n\}$.

- (i) show that $1 \in S$
- (ii) Show that S is inductive (that is, for all n , if $n \in S$, then $n+1 \in S$.)
- (iii) By the PMI, $S = N$.

Ex: Prove that

$$1 + 3 + 5 + \dots + (2n-1) = n^2$$

Proof: Define $S = \{n \in \mathbb{N} : 1 + 3 + \dots + (2n-1) = n^2\}$

(i) Since $1 = 1^2$, then $1 \in S$.

(ii) Let n be any natural number such that $n \in S$. Then $1 + 3 + 5 + \dots + (2n-1) = n^2$.

It follows that

$$\begin{aligned} & \underbrace{1 + 3 + 5 + \dots + 2n-1 + 2(n+1)-1}_{\text{---}} \\ &= n^2 + 2(n+1) - 1 \\ &= n^2 + 2n + 2 - 1 \\ &= n^2 + 2n + 1 \\ &= (n+1)^2 \end{aligned}$$

Thus, $n+1 \in S$. By, PMT, It follows that $S = \mathbb{N}$.

Thus $1 + 3 + 5 + \dots + 2n-1$ is true for all $n \in \mathbb{N}$.

Proof of $(\forall n \in \mathbb{N}) P(n)$ by Induction

(i) <Basic step> show that $P(1)$ is true

(ii) <Inductive step> show $P(n) \Rightarrow P(n+1)$:

(suppose that $P(n)$ is true for some $n \in \mathbb{N}$ and show that $P(n+1)$ is true

(iii) <conclusion> By steps (i) and (ii) and the PMI, $P(n)$ is true for all n

Ex: $(\forall n \in \mathbb{N}) (n+3 < 5n^2)$

Proof: (i) $1+3 < 5(1)^2 \Rightarrow 3 < 5$ true ✓

(ii) assume $n+3 < 5n^2$

$$\begin{aligned}(n+1)+3 &= (n+3)+1 < 5n^2+1 < 5n^2+10n+5 \\ &= 5(n+1)^2\end{aligned}$$

Thus

$$(n+1)+3 < 5(n+1)^2$$

(iii) By PMI, $n+1 < 5n^2$ for all $n \in \mathbb{N}$.

Thm 2.11 Archimedian Principle

For all natural numbers a, b $\exists s$ such that $a < sb$.

Generalized Principle of Math Induction

Let k be a natural number. If S is a subset of N with these two properties:

(i) $k \in S$

(ii) $(\forall n \in N) (n \geq k, n \in S \Rightarrow n+1 \in S)$

then S contains all natural numbers greater than or equal to k .

EX: Show $n^2 - n - 20 > 0$ for $n > 5$