

2.5 Equivalent Forms of Induction

The Principle of Complete Induction (PCI)

Suppose S is a subset of N with this property:
For all natural numbers n , if $\{1, 2, 3, \dots, n-1\} \subseteq S$,
then $n \in S$. Then $S = N$.

Thm: PMI is equivalent to PCI.

Ex: Every natural number $n > 1$ has a prime factor.

○ Proof:

Let S be $\{n \in N : n > 1 \text{ and } n \text{ has a prime factor}\}$. Notice 1 is not in S , but $2 \in S$. Let m be a natural number greater than 1 . Assume that for all $k \in \{2, 3, \dots, m-1\}$, $k \in S$. We must show that $m \in S$. If m has no factors other than 1 and itself, then it has a prime factor, itself. Suppose m has a factor x other than 1 and itself. It follows that $1 < x < m$. Thus, $x \in S$ and x has a prime factor. Thus, m has a prime factor. Thus, $m \in S$. Therefore $S = \{n \in N, n > 1\}$, and every natural number greater than 1 has a prime factor.

Fibonacci numbers

$$f_1 = 1, f_2 = 1, f_n = f_{n-1} + f_{n-2}, n \geq 3.$$

So we have 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ...

Ex: Let α be the positive solution to the equation $x^2 = x + 1$. [$\alpha = \frac{1+\sqrt{5}}{2} \approx 1.618$].

Prove that $f_n \leq \alpha^{n-1}$ for all $n \geq 1$.

Proof: Let m be a natural number and assume

for all $k \in \{1, 2, \dots, m-1\}$ that $f_k \leq \alpha^{k-1}$. Note the

special cases $m=1$ and $m=2$.

$$1 = f_1 \leq \alpha^0 = 1 \quad \text{and} \quad 1 = f_2 \leq \alpha^1 = \frac{1+\sqrt{5}}{2}$$

For $m \geq 3$,

$$\begin{aligned} f_m &= f_{m-1} + f_{m-2} \\ &\leq \alpha^{m-2} + \alpha^{m-3} \quad (\text{Induction hypothesis}) \\ &= \alpha^{m-3}(\alpha + 1) \quad (\text{since } \alpha + 1 = \alpha^2) \\ &= \alpha^{m-3} \alpha^2 \\ &= \alpha^{m-1} \end{aligned}$$

Therefore, $f_m \leq \alpha^{m-1}$. By PCI, we conclude that $f_n \leq \alpha^{n-1}$ for all $n \geq 1$.

The Well Ordering Principle (WOP)

Every nonempty subset of N has a smallest element.

Thm: WOP is equivalent to PMI.

Note: WOP, PMI, or PCI can all be used to show any property of the natural numbers (or a subset).

Ex: Every natural number larger than 1 has a prime factor.

Proof: If n is prime, then n is a prime factor of n . If n is composite, then n has factors other than 1 and n . Let p be the smallest of these factors. We must show that p is prime.

Suppose p is composite. Then p has a divisor d , with $1 < d < p$. Since d divides p and p divides n , then d divides n . But $d < p$, and this contradicts the def of p . Therefore p is prime.

Thm: The Division Algorithm for N .

Let $a \in N$ and $b \in N$ with $b \leq a$.
Then there exist $q \in N$ and $r \in N \cup \{0\}$ such
that $a = qb + r$, where $0 \leq r < b$.

The numbers q and r are the quotient and remainder, respectively.

Proof: Let $T = \{s \in N, a < sb\}$. By the
Archimedean Principle, this set is nonempty.
Therefore, by the WOP, T contains a smallest element
 w . Let $q = w - 1$ and $r = a - qb$.

Since $a \geq b$, then $w > 1$ and $q \in N$.

Since $q < w$, $a \geq qb$ and $r \geq 0$.

By def of r , $a = qb + r$.

Suppose $r \geq b$. Then $a - (w-1)b \geq b$, so $a \geq wb$.

This contradicts the fact that $a < wb$.

Therefore, $r < b$.

Thm: For natural numbers a and b , the GCD of
 a and b may be written as a linear combination
of a and b . That is, for $d = \text{GCD}(a, b)$, there
exist integers x & y such that $ax + by = d$.

Ex: $a = 85$ $b = 74$ $\text{GCD} = 1$.

So $85x + 74y = 1$ for some x & y .

$$85(27) + 74(-31) = 1$$

Try: $a = 24$, $b = 40$ $\text{GCD} = 8$.

$$24x + 40y = 8$$

$$24(2) + 40(-1) = 8$$