

2.6 Principles of Counting

- - Counting the number of elements in a set.
- A set A is finite iff $A = \emptyset$ or A has n elements for some $n \in \mathbb{N}$.
- The number of elements in A is denoted $\overline{A}$

Ex:

$$\overline{\{a, b\}} = 2$$

$$\overline{\{1, 2, 3, 5, 6\}} = 5$$

$$\overline{\emptyset} = 0$$

○ Thm: The sum rule.

If A and B are disjoint finite sets with $\overline{A} = m$ and $\overline{B} = n$, then $\overline{A \cup B} = m + n$.

Thm: Let $\mathcal{A} = \{A_i : i = 1, 2, \dots, n\}$ be a family of sets that are pairwise disjoint. Define $\overline{A_i} = a_i$ for $1 \leq i \leq n$. It follows that

$$\overline{\bigcup_{i=1}^n A_i} = \sum_{i=1}^n a_i$$

If A and B are not disjoint, then determining the number of elements in $A \cup B$, then we must subtract elements in A and B that were counted twice. Ex:

$$A = \{1, 2, 3\}$$

$$B = \{2, 3, 4\}$$

$$A \cup B = \{1, 2, 3, 4\}$$

$$\text{Then } \overline{A \cup B} = \bar{A} + \bar{B} - \overline{A \cap B} = 3 + 3 - 2 = 4.$$

Thm: For finite sets A and B,

$$(a) \overline{A \cup B} = \bar{A} + \bar{B} - \overline{A \cap B}$$

$$(b) \overline{A \cap B} = \bar{A} + \bar{B} - \overline{A \cup B}$$

What about more than 2 sets?

Use the Principle of Inclusion and Exclusion

Just like above, we must subtract (exclude) elements that were counted more than once. When dealing with 3 or more events, we must add in elements when too many are subtracted. For example:

$$\overline{A \cup B \cup C} = \bar{A} + \bar{B} + \bar{C} - [\overline{A \cap B} + \overline{A \cap C} + \overline{B \cap C}] + \overline{A \cap B \cap C}$$

Note: It is possible to do it algebraically:
using the rule $\overline{A \cup B} = \bar{A} + \bar{B} - \overline{A \cap B}$

$$\underbrace{\overline{A \cup B \cup C}}_D = \overline{D \cup C} = \bar{D} + \bar{C} - \overline{D \cap C}$$

$$= \overline{A \cup B} + \bar{C} - \overline{(A \cup B) \cap C}$$

$$= \bar{A} + \bar{B} - \overline{A \cap B} + \bar{C} - \overline{(A \cap C) \cup (B \cap C)}$$

$$= \bar{A} + \bar{B} + \bar{C} - \overline{A \cap B} - [\overline{A \cap C} + \overline{B \cap C} - \overline{A \cap B \cap C}]$$

$$= \bar{A} + \bar{B} + \bar{C} - \overline{A \cap B} - \overline{A \cap C} - \overline{B \cap C} + \overline{A \cap B \cap C}$$

For 4 sets:

$$\overline{A \cup B \cup C \cup D} = [\bar{A} + \bar{B} + \bar{C} + \bar{D}]$$

$$- [\overline{A \cap B} + \overline{A \cap C} + \overline{A \cap D} + \overline{B \cap C} + \overline{B \cap D} + \overline{C \cap D}]$$

$$+ [\overline{A \cap B \cap C} + \overline{A \cap B \cap D} + \overline{B \cap C \cap D} + \overline{A \cap C \cap D}]$$

$$- [\overline{A \cap B \cap C \cap D}]$$

Thm: The Product rule

If two independent tasks T_1 & T_2 are to be performed, and T_1 can be performed in n ways, and T_2 in m ways, then the two tasks can be performed in nm ways.

Thm: If k independent tasks, $T_1, \dots, T_k$ can be performed each in $n_1, \dots, n_k$ ways, respectively, then together they can be performed in

$$\prod_{i=1}^k n_i = n_1 \cdot \dots \cdot n_k$$

EX: Burger King:

6 combos $\rightarrow$ 6 different burgers
3 different sizes of fries
7 different flavors of soda.

How many different "combos" can you get?

$$6 \times 3 \times 7 = 126$$

DEF: A permutation of a set with n elements is an arrangement of the elements of the set in a specific order.

Recall $n! = \begin{cases} 1, & \text{if } n=0 \\ n \times (n-1)!, & \text{if } n>0. \end{cases}$

Thm: The number of permutations of a set of n elements is $n!$ $\rightarrow$ proof uses basic counting.

EX: How many permutations are there of the symbols $\diamond \circ \square$? $\underline{3 \times 2 \times 1 = 6}$

So $\diamond \square \circ$

$\diamond \circ \square$

$\square \circ \diamond$

$\square \diamond \circ$

$\circ \square \diamond$

$\circ \diamond \square$

6 of them!

Permutation Rule

The number of permutations of any r distinct objects taken from a set of n objects is

$$\frac{n!}{(n-r)!}$$

Ex: Awarding prizes at a contest.

Suppose only the top 5 in a sporting event are awarded prizes (and the order of the prizes are important because they are different). How many ways can 5 prizes be awarded if there is 15 contestants?

$$\frac{15 \cdot 14 \cdot 13 \cdot 12 \cdot 11}{(15-5)!} = \frac{15!}{10!} = 360,360$$

Some times the awards may be the same. Therefore, the order doesn't matter in which they are awarded. In this case, a combination of elements are taken. Note that in a combination of elements, the collection of objects is important, but not the order. For example,

Recall



What do each of the permutations contain?

→ 1 square, 1 circle, 1 diamond

→ same

→ same

→ same

→ same

→ same.

So there is only 1 combination of those elements. Here is another example.

A B C D. → draw 2 letters at a time

12 permutations What do each of the permutations contain?

A B B A → 1 A, 1 B

A C C A → 1 A, 1 C

A D D A → 1 D, 1 A

B C C B → 1 B, 1 C

B D D B → 1 B, 1 D

C D D C → 1 C, 1 D

So there are only 6 combinations

DEF: A combination of n elements taken r at a time is the selection of an r -element subset from an n -element set. It is denoted as $\binom{n}{r}$ and is read as "n choose r" or "n binomial r". The number $\binom{n}{r}$ is called a binomial coefficient.

Thm⁴ The Combination Rule

Let n be a pos. integer and r an integer $0 \leq r \leq n$. Then

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

EX: $A = \{2, 3, 6, 18, 39, 81, 442, 468, 574, 608\}$

a) In how many ways can 4 elements be selected from A so that their sum is less than 4 hundred?

There are 6 elements less than 100. So the number of ways is

$$\binom{6}{4} = \frac{6!}{2!4!} = \frac{6 \cdot 5}{2 \cdot 1} = 15 \text{ ways.}$$

Thm 2.24

Let n be a pos integer and r an integer such that $0 \leq r \leq n$.

(a) For $a, b \in \mathbb{R}$, $(a+b)^n = \sum_{r=0}^n \binom{n}{r} a^r b^{n-r}$

(b) $\sum_{r=0}^n \binom{n}{r} = 2^n$

(c) $\binom{n}{r} = \binom{n}{n-r}$

(d) $\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}$ for $r \geq 1$,
and $r \leq n-1$

Pascals triangle.