

Chapter 3 Relations

3.1 Cartesian Products and Relations

Ordered pairs $\rightarrow (a, b)$

ordered n -tuples $\rightarrow (a_1, a_2, \dots, a_n)$

Ex: $(2, 5) \neq (5, 2)$ (The order matters)

In fact, the only way they're equal is if they are all the same!

$$(a_1, \dots, a_n) = (x_1, \dots, x_n) \text{ iff } a_i = x_i \text{ } i=1, \dots, n.$$

DEF: let A and B be sets. The set of all ordered pairs having first coordinate in A and second coordinate in B is called the Cartesian product (cross product) of A and B and is written $A \times B$. Thus

$$A \times B = \{ (a, b) : a \in A \text{ and } b \in B \}$$

Ex: $A = \{1, 2\}$ $B = \{2, 3, 4\}$

$$A \times B = \{ (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4) \}$$

$$B \times A = \{ (2, 1), (3, 1), (4, 1), (2, 2), (3, 2), (4, 2) \}$$

Note that if $\bar{A} = m$ and $\bar{B} = n$, then $\overline{A \times B} = mn$

Thm: If A, B, C, D are sets, then

$$(a) \quad A \times (B \cup C) = (A \times B) \cup (A \times C)$$

$$(b) \quad A \times (B \cap C) = (A \times B) \cap (A \times C)$$

$$(c) \quad A \times \emptyset = \emptyset$$

$$(d) \quad (A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$$

$$(e) \quad (A \times B) \cup (C \times D) \subseteq (A \cup C) \times (B \cup D)$$

$$(f) \quad (A \times B) \cap (B \times A) = (A \cap B) \times (A \cap B)$$

Note that (e) is only showing a subset! There exists sets A, B, C, D , such that

$$(A \cup C) \times (B \cup D) \not\subseteq (A \times B) \cup (C \times D)$$

Proof of (a): (from the book)

The ordered pair $(x, y) \in A \times (B \cup C)$

iff $x \in A$ and $y \in B \cup C$

iff $x \in A$ and ($y \in B$ or $y \in C$)

iff $(x \in A \text{ and } y \in B) \text{ or } (x \in A \text{ and } y \in C)$

iff $(x, y) \in (A \times B) \text{ or } (x, y) \in (A \times C)$

iff $(x, y) \in (A \times B) \cup (A \times C)$

Thus $A \times (B \cup C) = (A \times B) \cup (A \times C)$

Relations

DEF

Let A and B be sets. R is a relation from A to B iff R is a subset of $A \times B$. If $(a, b) \in R$, we write aRb and say a is R -related (or simply related) to b . If $(a, b) \notin R$, we write $a \not R b$. A relation from A to A is called a relation on A .

EX: Let $A = \{1, 2, 3, 4\}$
 $B = \{-1, 1, 2, 4, 5\}$

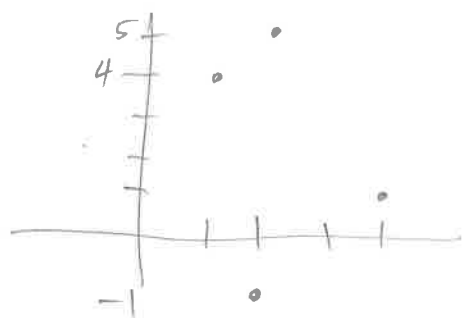
explicit list

$$R = \{(1, 4), (2, 5), (2, -1), (4, 1)\}$$

Table form:

1	4
2	5
2	-1
4	1

$$R = \{(x, y) \in A \times B : |x - y| = 3\}$$



Ex: Telephone (old one)

1	ABC 2	DEF 3
GHI 4	JKL 5	MNO 6
PRS 7	TUV 8	WXY 9
*	0	#

Note: Relation between

$\Delta = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and

$\Gamma = \{A, B, C, D, E, F, G, H, I, J, K, L, M, N, O, P, Q, R, S, T, U, V, W, X, Y, Z\}$

$(4, G) \in R$, $(3, T) \notin R$

$4 R G$

$3 \not R T$

Note that Q & Z aren't related to any number!

DEF: The domain of the relation R from A to B is the set

$$\text{Dom}(R) = \{x \in A : \exists y \in B \rightarrow x R y\}$$

The range of the relation R is the set

$$\text{Rng}(R) = \{y \in B : \exists x \in A \rightarrow x R y\}$$

The domain is the set of all first coordinates
and the range is " " " " second "

Ex: $A = \{s, p, q, \{r\}\}$

$$B = \{a, b, c, d\}$$

$$R = \{(p, a), (q, b), (q, c), (\{r\}, b)\}$$

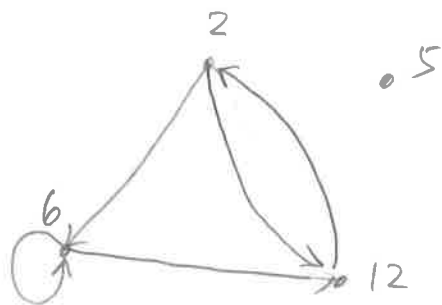
$$\text{Dom}(R) = \{p, q, \{r\}\}$$

$$\text{Rng}(R) = \{a, b, c\}$$

Ex: The directed graph.

$$S = \{(6, 12), (2, 6), (2, 12), (6, 6), (12, 2)\}$$

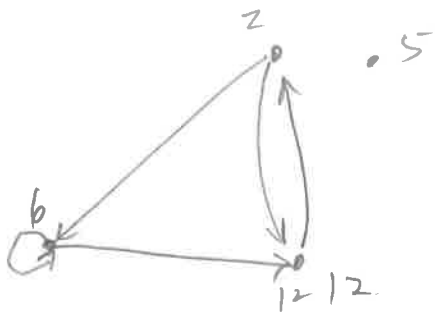
$$V = \{2, 5, 6, 12\}$$



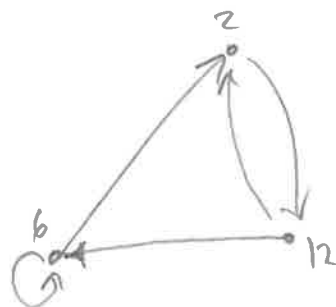
DEF: If R is a relation from A to B , then the inverse of R is $R^{-1} = \{(y, x) : (x, y) \in R\}$

Essentially, it is a swap of x & y . Ex:

$$S^{-1} = \{(12, 6), (6, 2), (12, 2), (6, 6), (2, 12)\}$$



S



S^{-1}

Thm: Let R be a relation from A to B .

(a) R^{-1} is a relation from B to A .

(b) $\text{Dom}(R^{-1}) = \text{Rng}(R)$

(c) $\text{Rng}(R^{-1}) = \text{Dom}(R)$

Proof of A: Suppose $(x, y) \in R^{-1}$, then $(y, x) \in R$.

Since R is a relation from A to B , $R \subseteq A \times B$. Thus, $y \in A$ and $x \in B$. Therefore, $(x, y) \in B \times A$. Thus, $R^{-1} \subseteq B \times A$, which means that R^{-1} is a relation from B to A .

DEF: Let R be a relation from A to B , and let S be a relation from B to C . The composite of R and S is

$$S \circ R = \{(a, c) : \text{there exists } b \in B \text{ such that } (a, b) \in R \text{ and } (b, c) \in S\}$$

$S \circ R$ is a relation from A to C (since $S \circ R \subseteq A \times C$)

Note: To determine $S \circ R$, first apply R , then S .

Ex:

$$A = \{1, 2, 3, 4, 5\}$$

$$B = \{p, q, r, s, t\}$$

$$C = \{x, y, z, w\}$$

$$R = \{(1, p), (1, q), (2, q), (3, r), (4, s)\} \text{ (from } A \text{ to } B)$$

$$S = \{(p, x), (q, x), (q, y), (s, z), (t, z)\}$$

Thm: Suppose A, B, C, D are sets.

Let R be a relation from A to B ,
 S be a relation from B to C
 T be a relation from C to D

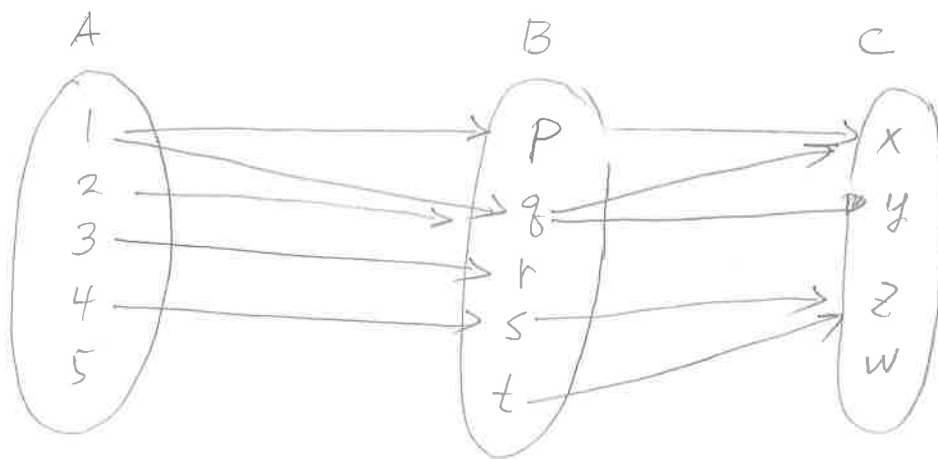
(a) $(R^{-1})^{-1} = R$.

(b) $T \circ (S \circ R) = (T \circ S) \circ R$ (associative)

(c) $I_B \circ R = R$ and $R \circ I_A = R$

(d) $(S \circ R)^{-1} = R^{-1} \circ S^{-1}$

Note: relational databases are an example of an implementation of relationships.



$$S \circ R = \{(1, x), (1, y), (2, x), (2, y), (4, z)\}$$

EX: Let $R = \{(x, y) \in \mathbb{R} \times \mathbb{R} : y = x + 1\}$

$$S = \{(x, y) \in \mathbb{R} \times \mathbb{R} : y = x^2\}$$

$$R \circ S = \{(x, y) : (x, z) \in S \text{ and } (z, y) \in R \text{ for some } z \in \mathbb{R}\}$$

$$= \{(x, y) : z = x^2 \text{ and } y = z + 1 \text{ for some } z \in \mathbb{R}\}$$

$$= \{(x, y) : y = x^2 + 1\}$$

$$S \circ R = \{(x, y) : (x, z) \in R \text{ and } (z, y) \in S \text{ for some } z \in \mathbb{R}\}$$

$$= \{(x, y) : z = x + 1 \text{ and } y = z^2 \text{ for some } z \in \mathbb{R}\}$$

$$= \{(x, y) : y = (x + 1)^2\}$$

Note: $S \circ R \neq R \circ S$.