

3.2 Equivalence Relations

DEF: Let A be a set and R be a relation on A .

R is reflexive on A iff $(\forall x \in A)(x R x)$

R is symmetric iff $(\forall x, y \in A)(x R y \Rightarrow y R x)$

R is transitive iff $(\forall x, y, z \in A)(x R y \text{ and } y R z \Rightarrow x R z)$

Ex: $B = \{2, 5, 6, 7\}$

$$S_1 = \{(2, 5), (5, 6), (2, 6)\}$$

$$S_2 = \{(2, 2), (2, 5)\}$$

$$S_3 = \{(2, 6), (5, 6)\}$$

Note: None of these relations are reflexive on B or symmetric.

But for all of them, they are transitive.

Note for S_3 if $x S_3 y$ and $y S_3 z$, then $x S_3 z$ is true because the antecedent is false.

Ex: Let R be the relation "is a subset of" on $\mathcal{P}(\mathbb{Z})$ the power set of $\mathbb{Z}$. R is reflexive on $\mathcal{P}(\mathbb{Z})$ since every subset is a subset of itself. R is Transitive by Thm 2.3. Note finally since $\{1\} \subseteq \{1, 2\}$, but $\{1, 2\} \not\subseteq \{1\}$, it follows that R is not reflexive.

Ex: Let $STNR$ designate the relation

$$\{(x, y) \in \mathbb{Z} \times \mathbb{Z} : xy > 0\} \text{ on } \mathbb{Z}.$$

Since $x STNR x$ is true for all x except 0, it follows that $STNR$ is not reflexive.

Note $STNR$ is symmetric, since $xy > 0 \Rightarrow yx > 0$. $STNR$ is also transitive. Verify!

Suppose $x STNR y$ and $y STNR z$. It follows that $xy > 0$ and $yz > 0$. If y is positive, then so is x and z . It follows that $xz > 0$ ($x STNR z$). If y is negative, then both x and z are negative. Thus, $xz > 0$ also ($x STNR z$).

Hence, the name $STNR$

symmetric, transitive, not reflexive.

a directed graph can illustrate these properties nicely.

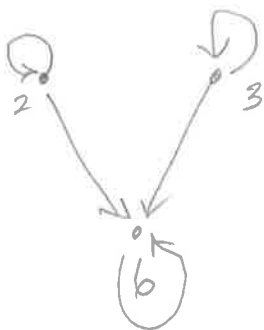
A relation is reflexive iff every vertex of the digraph has a loop. (an edge from the vertex to itself.)

A relation is symmetric iff between any two vertices, there are either no edges, or an edge in both directions.

A relation is transitive iff whenever there is an edge from x to y and y to z , there must be an edge from x to z .

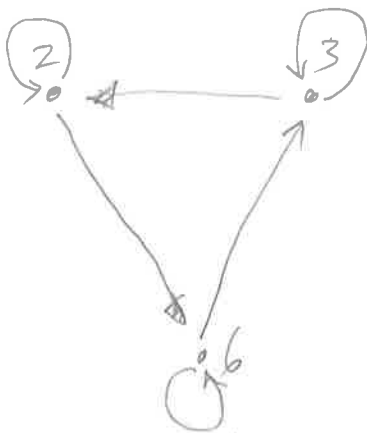
EX:

"divides"



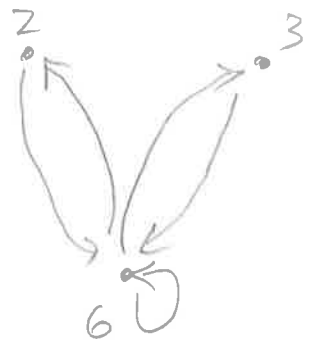
reflexive
not symmetric
transitive

$\geq$



reflexive
not symmetric
transitive

xSy iff $x + y > 7$



not reflexive
symmetric
not transitive

DEF: A relation R on a set A is an equivalence relation on A iff R is reflexive, symmetric, and transitive.

Ex: L = relation on the set of all people such that xLy iff x & y have the same last name. (equivalence relation)

Def: Let R be an equivalence relation on A .

For $x \in A$, the equivalence class of x determined by R is the set.

$$x/R = \{y \in A : xRy\}$$

This is read "the class of x modulo R " or " $x \bmod R$ ". The set of all equivalence classes is called A modulo R and is denoted $A/R = \{x/R : x \in A\}$.

Ex:

Thm 3.4 The relation $\equiv_m$ is an

equivalence relation on the integers. Further
more $\mathbb{Z}_m$ has m distinct elements, $\bar{0}, \bar{1},$
 $\bar{2}, \dots, \overline{m-1}$