

3.3 Partitions

Partitioning — zip codes
state boundaries
time zones

DEF: Let A be a nonempty set. $\mathcal{A}$ is a partition of A iff $\mathcal{A}$ is a set of subsets of A such that

- (i) If $X \in \mathcal{A}$, then $X \neq \emptyset$.
- (ii) If $X \in \mathcal{A}$, and $Y \in \mathcal{A}$, then $X=Y$ or $X \cap Y = \emptyset$
- (iii) $\bigcup_{X \in \mathcal{A}} X = A$.

EX: Partitions:

$$\begin{aligned} \mathbb{R} &= \{Q, \mathbb{R} - Q\} \\ &= \bigcup_{k=1}^{\infty} G_k \Rightarrow \text{where } G_k = [k, k+1) \end{aligned}$$

$$\begin{aligned} \mathbb{Z} &= \{E, O\} \quad (\text{even and odd integers}) \\ &= \{\mathbb{N}, \mathbb{Z}^-\} \quad \mathbb{Z}^- \text{ is the set of neg int.} \\ &= \{\{0\}, \{1, -1\}, \{2, -2\}, \{3, -3\}, \dots\} \end{aligned}$$

Ex:

$W = \{ \text{Fred, Ethel, Squiggy, Lenny, Norm, Carla} \}$

$C = \{ \text{blue, green, red, white} \}$

$B_c = \{ x \in W : x \text{ is wearing clothing with color } C \}$

$\mathcal{B} = \{ B_c : c \in C \} \rightarrow \text{probably not a partition.}$

Why?

- If Fred is wearing blue and green, he belongs to at least 2 sets in $\mathcal{B}$ (ii violated)
- If Carla is only wearing brown, she does not belong to any set in $\mathcal{B}$ (iii violated)
- If no one wears red, then at least one set in $\mathcal{B}$ is empty (i violated).

Thm 3.5 Let R be an equivalence relation on a nonempty set A . Then

(a) For all $x \in A$, $x/R \subseteq A$ and $x \in x/R$ (Thus $x/R \neq \emptyset$)

(b) $\bigcup_{x \in A} x/R = A$

(c) $x R y$ iff $x/R = y/R$

(d) $x \not R y$ iff $x/R \cap y/R = \emptyset$

Thus the set $\{ x/R : x \in A \}$ of equiv. classes is a partition.

Thm 3.5 says that every equiv. relation on a set determines a partition on the set. The partition is the set of equiv. classes.

Ex:

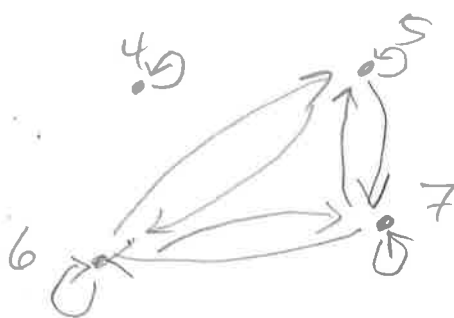
$$A = \{4, 5, 6, 7\}$$

$$T = \{(5, 7), (7, 6), (6, 6), (5, 5), (7, 7), (7, 5), (6, 5), (5, 6), (6, 7), (4, 4)\}$$

We first find the equiv classes:

$$4/T = \{4\}$$

$$5/T = \{5, 6, 7\}$$



so $A/T = \{\{4\}, \{5, 6, 7\}\}$ is the partition.

Note that every partition determines an equivalent class or vice versa.

Thm

Let $\mathcal{B}$ be a partition of the non-empty set A

for x and $y \in A$, define $x Q y$ iff there exists $C \in \mathcal{B}$ such that $x \in C$ and $y \in C$. Then

(a) Q is an equivalence relation on A

(b) $A/Q = \mathcal{B}$

Ex: Let $A = \{1, 2, 3, 4\}$

$\mathcal{B} = \{\{1\}, \{2, 3\}, \{4\}\}$ be a partition of A .

The equivalence relation is

$$\{(1, 1), (2, 2), (3, 3), (4, 4), (2, 3), (3, 2)\}$$

The three equiv. classes are

$$1/Q = \{1\}$$

$$2/Q = 3/Q = \{2, 3\}$$

$$4/Q = \{4\}$$

Note the set of equivalence classes is $\mathcal{B}$.

Ex: The set $\mathcal{A} = \{A_0, A_1, A_2, A_3\}$ is a partition of $\mathbb{Z}$, where

$$A_0 = \{4k : k \in \mathbb{Z}\}$$

$$A_1 = \{4k+1 : k \in \mathbb{Z}\}$$

$$A_2 = \{4k+2 : k \in \mathbb{Z}\}$$

$$A_3 = \{4k+3 : k \in \mathbb{Z}\}$$

The integers x & y are in the same set iff
 $x = 4n+i$ and $y = 4m+i$, or if $x-y$ is a multiple of 4. Note: each A_i is the residue class of i modulo 4. for $i = 0, 1, 2, 3$.