

4.1 Functions as Relations.

DEF A function (or mapping) from A to B is a relation f from A to B such that

- (i) the domain of f is A .
- (ii) If $(x, y) \in f$ and $(x, z) \in f$, then $y = z$

we write $f: A \rightarrow B$ and this is read "f is a function from A to B ," or "f maps A to B ."

The set B is called the codomain of f . In the case where $A = B$, we say that f is a function on A .

No restriction is placed on A or B . They can be sets of
numbers
ordered pairs
functions
sets of sets of functions.

EX Let $A = \{1, 2, 3\}$ $B = \{4, 5, 6\}$. All the sets

$R_1 = \{(1, 4), (2, 5), (3, 6), (2, 6)\} \rightarrow$ not a function (ii) violated

$R_2 = \{(1, 4), (2, 6), (3, 5)\} \rightarrow$ a function from A to B

$R_3 = \{(1, 5), (2, 5), (3, 4)\} \rightarrow$ a function from A to B

$R_4 = \{(1, 4), (3, 6)\} \rightarrow$ not a function from A to B .

But is a function from $\{1, 3\}$ to B .

Note: Range and Codomain are not the same.

For example,

$$\text{Rng}(f_4) = \{4, 6\} \neq \underbrace{\{4, 5, 6\}}_{\text{codomain}}$$

Note that these defs are not strict since some mathematicians call B the range, even when B is not the set of second coordinates, and even when A is not the $\text{Dom}(f)$, but rather a proper subset of A .

○ DEF: Let $f: A \rightarrow B$. We write $y = f(x)$ when $(x, y) \in f$. We say that y is the value of f at x (or image of f at x) and that x is a pre-image of y under f .

Ex: Let $F = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} : y = x^2\}$, then F is a function with domain $\mathbb{Z}$ and $F(x) = x^2$. The image of 4 is 16, $F(-3) = 9$, $F(t+2) = (t+2)^2$ and the value of F at 10 is 100. Both 5 and -5 are preimages of 25. Since 7 has no preimage in $\mathbb{Z}$, 7 is not in the range of F . The $\text{Rng}(F)$ is $\{0, 1, 4, 9, 16, \dots\}$

Ex: Let $\text{csc} = \{ (x, y) \in \mathbb{R} \times \mathbb{R} : y = \frac{1}{\sin x} \}$.

Since $\sin x = 0$ for $x = k\pi, k \in \mathbb{Z}$ then

$$\text{Dom}(\text{csc}) = \mathbb{R} - \{ k\pi : k \in \mathbb{Z} \}$$

The image of $\frac{\pi}{4}$ is $\text{csc}\left(\frac{\pi}{4}\right) = \frac{1}{\sin \frac{\pi}{4}} = \frac{1}{\frac{1}{\sqrt{2}}} = \sqrt{2}$

A pre-image of 2 is $\frac{\pi}{6}$ and the set of pre-images of 2 is $\{ \frac{\pi}{6} + 2\pi k : k \in \mathbb{Z} \} \cup \{ \frac{5\pi}{6} + 2\pi k : k \in \mathbb{Z} \}$. $\text{Rng}(\text{csc}) = (-\infty, -1] \cup [1, \infty)$

Ex: The empty function ϕ is a function on the empty set $\emptyset$. If $f: A \rightarrow B$ and any one of f , A , or B are empty, then all three are empty.

Ex: Identity function on A

$$I_A(x) = x \quad \text{Dom}(I_A) = A \quad \text{Rng}(I_A) = A$$

Ex:

If $A \subseteq B$, then $i: A \rightarrow B$ given by
 $i(x) = x$ for every $x \in A$
 i is the inclusion map from A to B .

Ex:

Let $k: \mathbb{R} \rightarrow \mathbb{R}$ $k(x) = 3$ for every $x \in \mathbb{R}$.

Then $\text{Rng}(k) = \{3\}$

k is a constant function.

Ex: Assume that a universe has been specified,
and that $A \subseteq U$ Def: $\chi_A: U \rightarrow \{0, 1\}$

$$\chi_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \in U - A \end{cases}$$

χ_A = characteristic function of A .

Ex:

Generalization of the characteristic function:

The Step function

$\mathcal{C} = \{C_\delta : \delta \in \Delta\}$ partitioning
a set A . for each $\delta \in \Delta$, b_δ is in the set B .

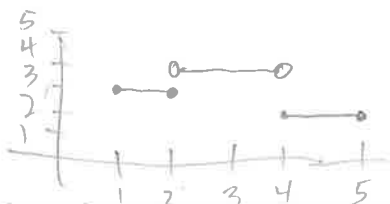
Define $f: A \rightarrow B$ by for $x \in C_\delta$, let $f(x) = b_\delta$

$$A = [1, 5] \quad C_1 = [1, 2] \quad C_2 = (2, 4), \quad C_3 = [4, 5]$$

$$b_1 = 3$$

$$b_2 = 4$$

$$b_3 = 2$$



$g = \text{Greatest integer function} = \text{floor function}$
 $\text{Dom}(g) = \mathbb{R} \quad \text{Codomain } \mathbb{Z}.$

Thm 4.1 Two functions f & g are
equal iff

- (i) $\text{Dom}(f) = \text{Dom}(g)$, and
- (ii) for all $x \in \text{Dom}(f)$, $f(x) = g(x)$.