

4.2 Constructions of functions

For a function $F: A \rightarrow B$, the inverse of F is the relation from B to A ,

$$F^{-1} = \{(x, y) : (y, x) \in F\}$$

$F: A \rightarrow B$ $G: B \rightarrow C$ The composite of F and G is the relation from A to C :

$$G \circ F = \{(x, z) \in A \times C : (x, y) \in F \text{ and } (y, z) \in G, \text{ for some } y \in B\}.$$

Note: Thm 4.2 says the composite of two functions is a function.

Ex: Let $F = \{(x, y) : y = 2x + 1\}$ and
 $G = \{(x, y) : y = x^2\}$

Domain & codomain are $\mathbb{R}$.

$$\begin{aligned} F^{-1} &= \{(x, y) \in \mathbb{R} \times \mathbb{R} : (y, x) \in F\} \\ &= \{(x, y) \in \mathbb{R} \times \mathbb{R} : x = 2y + 1\} \\ &= \{(x, y) \in \mathbb{R} \times \mathbb{R} : y = \frac{x-1}{2}\} \end{aligned}$$

and

$$\begin{aligned} G^{-1} &= \{(x, y) \in \mathbb{R} \times \mathbb{R} : (y, x) \in G\} \\ &= \{(x, y) \in \mathbb{R} \times \mathbb{R} : x = y^2\} \end{aligned}$$

Note: The inverse of F is a function, but
" " " G " not a function

since $(4, 2) \in G^{-1}$ and $(4, -2) \in G^{-1}$

The composite of F and G is

$$G \circ F = \{(x, z) \in \mathbb{R} \times \mathbb{R} : \exists y \in \mathbb{R} \text{ such that } (x, y) \in F \text{ and } (y, z) \in G\}.$$

$$= \{(x, z) \in \mathbb{R} \times \mathbb{R} : \exists y \in \mathbb{R} \text{ such that } y = 2x + 1 \text{ and } z = y^2\}$$

$$= \{(x, z) \in \mathbb{R} \times \mathbb{R} : z = (2x + 1)^2\}$$

○ we can also compute

$$F \circ F$$

$$F \circ G, \text{ etc.}$$

Note: $F \circ G \neq G \circ F$ (not always)

We can take advantage of the fact that each element of the domain of a function has a unique image to simplify the notation for composition. Let $H: A \rightarrow B$ and $K: B \rightarrow C$. Since $(x, y) \in H$ and $(y, z) \in K$ may be written in the form $y = H(x)$ and $z = K(y)$, we can write

$$z = K(H(x)) = (K \circ H)(x)$$

Ex: $H(x) = \sin x$
 $K(x) = x^2 + 6$
 $L(x) = e^x$

$$(H \circ K)(x) = H(K(x)) = H(x^2 + 6) = \sin(x^2 + 6)$$

$$(K \circ L)(x) = K(L(x)) = K(e^x) = (e^x)^2 + 6 = e^{2x} + 6$$

$$(L \circ H)(x) = L(H(x)) = L(\sin x) = e^{\sin x}$$

Ex: $Z_6 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}\}$ $Z_3 = \{[0], [1], [2]\}$

$$H: Z_6 \rightarrow Z_3 \rightarrow \begin{aligned} H(\bar{0}) &= [0], & H(\bar{1}) &= [1], & H(\bar{2}) &= [2] \\ H(\bar{3}) &= [0], & H(\bar{4}) &= [1], & H(\bar{5}) &= [2] \end{aligned}$$

$$K: Z_3 \rightarrow Z_3 \quad K([0]) = [0] \quad K([1]) = [2], \quad K([2]) = [1]$$

Then

$$(K \circ H)(\bar{0}) = K(H(\bar{0})) = K([0]) = [0]$$

$$(K \circ H)(\bar{4}) = K(H(\bar{4})) = K([1]) = [2]$$

K^{-1} is a function (Note $K^{-1} = K$) The composite

$K^{-1} \circ K$ has images

$$(K^{-1} \circ K)([0]) = K^{-1}([0]) = [0],$$

$$(K^{-1} \circ K)([1]) = K^{-1}([2]) = [1]$$

$$(K^{-1} \circ K)([2]) = K^{-1}([1]) = [2]$$

Thus $K^{-1} \circ K$ is the identity function on Z_3

Thm 4.2

Let A, B , and C be sets and

$$F: A \rightarrow B$$

$$G: B \rightarrow C$$

Then $G \circ F$ is a function from A to C and $\text{Dom}(G \circ F) = A$

Proof : Sect 3.1 proved $G \circ F$ is a relation from A to C . To show that $G \circ F$ is a function from A to C , let $(x, y) \in G \circ F$ and $(x, z) \in G \circ F$. We will show that $y = z$.

Since $(x, y) \in G \circ F$, there exists $u \in B$ such that $(x, u) \in F$ and $(u, y) \in G$. Likewise, there exists $v \in B$ such that $(x, v) \in F$ and $(v, z) \in G$. Since F is a function $(x, u) \in F$ and $(x, v) \in F$ imply that $u = v$. Since G is a function, $(u, y) \in G$, $(v, z) \in G$ and $u = v$, then $y = z$.

show $\text{Dom}(G \circ F) = A$ yourself!

Thm 4.3 Let A, B, C , and D be sets. and $f: A \rightarrow B$, $g: B \rightarrow C$, and $h: C \rightarrow D$. Then $(h \circ g) \circ f = h \circ (g \circ f)$. That is, the composition of functions is associative.

Thm 4.4 Let $f: A \rightarrow B$, Then $f \circ I_A = f$, $I_B \circ f = f$

Thm 4.5 Let $f: A \rightarrow B$ with $\text{Rng}(f) = C$. If

f^{-1} is a function, then $f^{-1} \circ f = I_A$ and $f \circ f^{-1} = I_C$

DEF: Let $f: A \rightarrow B$ and let $D \subseteq A$. The restriction of f to D is the function

$$f|_D = \{(x, y) : y = f(x) \text{ and } x \in D\}.$$

If g and h are functions and g is a restriction of h , we say h is an extension of g .

Thm 4.6 Let h & g be functions with $\text{Dom}(h) = A$ and $\text{Dom}(g) = B$. If $A \cap B = \emptyset$, then $h \cup g$ is a function with domain $A \cup B$. Furthermore,

$$(h \cup g)(x) = \begin{cases} h(x), & \text{if } x \in A \\ g(x), & \text{if } x \in B. \end{cases}$$

(Piecewise defined functions!)