

4.5

Sequences

(Not in 5th)

If S is a sequence, and $n \in \mathbb{N}$, the image of n is called the n^{th} term of S , and is usually written as S_n . The sequence is denoted by (S_n) or $(S_1, S_2, \dots)$

EX: $S = (1, 3, 5, 7, \dots)$

The n^{th} term is $S_n = 2n - 1 \Rightarrow (2n - 1)$

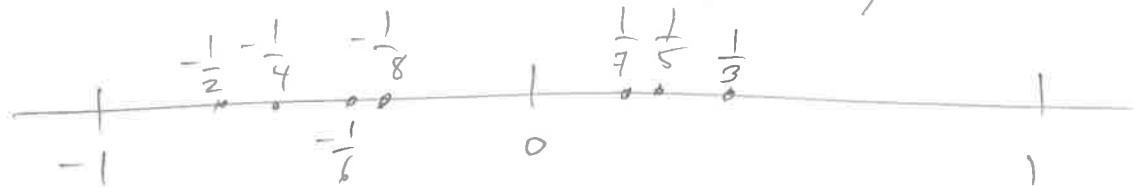
$(Y_n) \rightarrow$ sequence given by $(-1)^n$ has as its first few terms $-1, 1, -1, 1, \dots$

Fibonacci sequence

$$f_1 = 1, f_2 = 1, n > 2 \quad f_n = f_{n-1} + f_{n-2}$$

Ex: Ask: what happens to the value of the n^{th} term as you go further out in the sequence?

Try the sequence $x = \left(\frac{(-1)^n}{n+1} \right)$



DEF: A sequence x has a limit L iff for every real $\varepsilon > 0$, there exists a natural number N such that if $n > N$, then $|x_n - L| < \varepsilon$.

If the number L exists, we say x converges (to L) and write either

$$\lim_{n \rightarrow \infty} x_n = L \quad \text{or} \quad x_n \rightarrow L.$$

If no such number L exists, we say that x diverges.

In symbols $(\forall \varepsilon > 0)(\exists N \in \mathbb{N})(\forall n \in \mathbb{N})(n > N \Rightarrow |x_n - L| < \varepsilon)$

When using the definition to prove a limit, we have some preliminary work to do (scratch work). This scratch work is not the proof, but is necessary for the proof.

Ex: Prove that the sequence given by $x_n = \frac{(-1)^n}{n+1}$ converges to 0.

Prelim work
we want to find a relationship between n and ε .

Given $\varepsilon > 0$, we must find N such that if $n > N$, $|x_n - 0| < \varepsilon$. Since $|x_n - 0| = |x_n| = \left| \frac{(-1)^n}{n+1} \right| = \frac{1}{n+1}$, we need $\frac{1}{n+1} < \varepsilon \Rightarrow \frac{1}{\varepsilon} < n+1 \Rightarrow n > \frac{1}{\varepsilon} - 1$. This last inequality tells us how large N should be.

We now begin the proof!

Ex: Prove that $\left(\frac{(-1)^n}{n+1}\right)$ converges to 0.

Proof. Let $\varepsilon > 0$. Define N as any integer larger than $\frac{1}{\varepsilon} - 1$. Suppose $n > N$. Then it follows that

$$n > \frac{1}{\varepsilon} - 1 \Rightarrow n+1 > \frac{1}{\varepsilon} \Rightarrow \frac{1}{n+1} < \varepsilon.$$

Since $|x_n - 0| = \left| \frac{(-1)^n}{n+1} \right| = \frac{1}{n+1}$, then $|x_n - 0| < \varepsilon$.

Thus, $\lim_{n \rightarrow \infty} \frac{(-1)^n}{n+1} = 0$.

Ex: Prove that $\frac{(-1)^n}{n+1}$ does not converge to $\frac{1}{10}$.

Note that you need to prove a denial of the def of the limit; that is, find an $\varepsilon > 0$, such that $\forall N$, there is $n > N$ such that $|x_n - L| > \varepsilon$.

Proof: Let $\varepsilon = \frac{1}{100}$ and let N be a natural number. Let n be an odd integer greater than n . It follows that

$$x_n = \frac{-1}{n+1} < 0 \Rightarrow \left| x_n - \frac{1}{10} \right| > \frac{1}{10} > \varepsilon.$$

Therefore, x_n does not converge to $\frac{1}{10}$.

Thm: If a sequence x converges, then its limit is unique.

Proof: read it!

You can typically figure out limits using practice
For example,

$$x_n = \frac{\sin(\frac{1}{n})}{\frac{1}{n}} \Rightarrow x_5 = .993346, x_{20} = .995833, x_{60} = .999954$$

use this method with care!

Ex:

$$x_n = \frac{23^n}{n!} \Rightarrow$$

$$x_1 = 23, x_2 = 264.5, x_3 = 2621.83\bar{3}, x_4 = 11660.04$$

$$x_{22} = 807692035, x_{23} = 867692035, x_{24} = 774038200$$

$$x_{60} = .6074287, x_{90} = 2.418 \times 10^{-16}$$

Ex: Show $x_n = \frac{3n^2}{n^2+1}$ converges

Scratch work: I guess x_n converges to 3.

So $\left| \frac{3n^2}{n^2+1} - 3 \right| < \varepsilon$ is what is eventually wanted.

$$\text{So } \left| \frac{3n^2}{n^2+1} - 3 \right| = \left| \frac{3n^2 - 3(n^2+1)}{n^2+1} \right| = \left| \frac{\cancel{3n^2} - 3n^2 - 3}{n^2+1} \right| = \frac{3}{n^2+1}$$

$$\text{So } \frac{3}{n^2+1} < \varepsilon \Rightarrow n^2+1 > \frac{3}{\varepsilon} \Rightarrow n^2+1 > n > \frac{3}{\varepsilon}$$

So proceed with the proof:

Proof: Let $\varepsilon > 0$. Let N be a natural # greater than

$\frac{3}{\varepsilon}$. Suppose $n > N$. Then $n > \frac{3}{\varepsilon} \Rightarrow n^2+1 > n > \frac{3}{\varepsilon}$

$$\Rightarrow \frac{3}{n^2+1} < \varepsilon \Rightarrow \left| \frac{3n^2}{n^2+1} - 3 \right| < \varepsilon.$$

Therefore, $\frac{3n^2}{n^2+1} \rightarrow 3$ as $n \rightarrow \infty$.

Ex: Prove $y_n = (-1)^n$ diverges.

Proof: Suppose $\lim_{n \rightarrow \infty} y_n = L$, for $L \in \mathbb{R}$. Let $\varepsilon = 1$.

Let $N \in \mathbb{N}$. If $L > 0$, let n be any odd integer greater than N . Then it follows $y_n = -1$ and $1 - L < -1$

○ If $L \leq 0$, let n be any even integer greater than N ; then it follows that $y_n = 1$ and $1 - L > 1$. In both cases, we have shown that there is an $n > N$ and $|y_n - L| \geq 1$. Thus $y_n \not\rightarrow L$. Since this is true for all L , y_n diverges.

Thm 4.18 Sandwich Thm

Suppose a, b, c are real sequences such that
 $a_n \leq b_n \leq c_n$ for all $n \in \mathbb{N}$. If
 $a_n \rightarrow L$ and $c_n \rightarrow L$, then $b_n \rightarrow L$.

Proof: Suppose $a_n \rightarrow L$ and $c_n \rightarrow L$. Let $\varepsilon > 0$.

There exist natural numbers N_1 & N_2 such that

$$n > N_1 \Rightarrow |a_n - L| < \varepsilon \quad \text{and}$$

$$n > N_2 \Rightarrow |c_n - L| < \varepsilon.$$

Let N be the larger of N_1, N_2 . Since $a_n \leq b_n \leq c_n$,

$$a_n - L \leq b_n - L \leq c_n - L.$$

Therefore, for $n > N$,

$$-\varepsilon < a_n - L \leq b_n - L \leq c_n - L < \varepsilon.$$

Thus,

$$|b_n - L| < \varepsilon \quad \text{for all } n > N, \text{ so}$$

$$b_n \rightarrow L.$$

Ex: $x_n = \frac{\sin n}{n} \Rightarrow -1 \leq \sin n < 1$

$$\Rightarrow \underbrace{-\frac{1}{n}}_{a_n} \leq \frac{\sin n}{n} \leq \underbrace{\frac{1}{n}}_{c_n}$$

Since $a_n \rightarrow 0$ and $c_n \rightarrow 0$, then $b_n = \frac{\sin n}{n} \rightarrow 0$ also.