

5.2 Infinite Sets

Thm 5.11 The set N of natural numbers is infinite.

Proof: Let E^+ be the set of even positive integers. The function $f: N \rightarrow E^+$ defined by $f(x) = 2x$ is a 1-1 correspondence from N onto E^+ . Thus, $N \approx E^+$, one of its proper subsets. Therefore, N is infinite.

DEF: Let S be a set. S is denumerable iff $S \approx N$.

For a denumerable set S , we say that S has cardinal number $\aleph_0$ (or cardinality $\aleph_0$) and we write $\bar{S} = \aleph_0$.

Thm: The set Z of integers is denumerable

Proof: Find a 1-1 correspondence from N to Z .

EX: Let P be the set of reciprocals of pos. integer powers of 2.

Note

$$P = \left\{ \frac{1}{2^k} : k \in N \right\}$$

P is a function from N to P so P is denumerable

Ex: The set $K = \{p, q, r\} \cup \{n \in \mathbb{N} : n \neq 5\}$ is denumerable.

Proof:

$$g(n) = \begin{cases} p, & \text{if } n=1 \\ q, & \text{if } n=2 \\ r, & \text{if } n=3 \\ n-3, & \text{if } 4 \leq n \leq 7 \\ n-2, & \text{if } n \geq 8 \end{cases}$$

Thm: (a) The set $\mathbb{N} \times \mathbb{N}$ is denumerable

(b) If A, B are denumerable, then $A \times B$ is denumerable.

Proof: (a) We need to show that there is a bijection $f: \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$. Define

$$f(m, n) = 2^{m-1} (2n-1)$$

We showed before that f is 1-1 and onto before.

Therefore, $\mathbb{N} \times \mathbb{N}$ is denumerable.

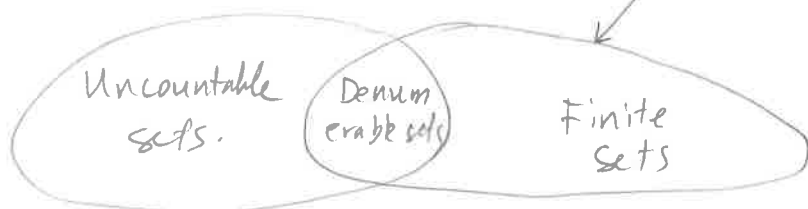
Proof (b)

Since $A \approx \mathbb{N}$, and $B \approx \mathbb{N}$, $A \times B \approx \mathbb{N} \times \mathbb{N}$.

Since $A \times \mathbb{N} \approx \mathbb{N}$, then $A \times B \approx \mathbb{N}$ and $A \times B$ is denumerable.

infinite sets

countable sets



DEF: A set S is countable iff S is finite or denumerable. S is uncountable iff it is not countable.

Review: any number in $(0,1)$ can be written as

$$0.a_1 a_2 a_3 \dots \quad \text{where } 0 \leq a_i \leq 9$$

$x = 0.a_1 a_2 \dots$ is said to be in normalized form iff there is no k such that for all $n > k$, $a_n = 9$.

Note that every real number can be expressed uniquely in normalized form.

Thm: The open interval $(0,1)$ is uncountable.

(We must show that $(0,1)$ is neither finite nor denumerable)

First, since $S = \{\frac{1}{2^k} : k \in \mathbb{N}\}$ is a subset of $(0,1)$, then $(0,1)$ must be infinite.

Assume $(0,1)$ is denumerable. Thus, there is a function $f: \mathbb{N} \rightarrow (0,1)$ that is 1-1 and onto $(0,1)$. Write the images of f , for each $n \in \mathbb{N}$ in normalized form:

$$f(1) = 0.a_{11} a_{12} a_{13} a_{14} a_{15} \dots$$

$$f(2) = 0.a_{21} a_{22} a_{23} a_{24} a_{25} \dots$$

$$f(3) = 0.a_{31} a_{32} a_{33} a_{34} a_{35} \dots$$

$$f(n) = 0.a_{n1} a_{n2} a_{n3} a_{n4} a_{n5} \dots$$

Let $b = 0.b_1 b_2 b_3 b_4 b_5 \dots$, where

$$b_i = \begin{cases} 5, & \text{if } a_{ii} \neq 5 \\ 3, & \text{if } a_{ii} = 5 \end{cases} \quad (\text{choice of 3 \& 5 is arbitrary}).$$

Note: $b \in (0,1)$ because of how it is constructed.

However, for each n , b differs from $f(n)$ in the n^{th} decimal place. Thus $b \neq f(n)$ for any $n \in \mathbb{N}$, which means $b \notin \text{Rng } f$. Thus, f is not onto $(0,1)$. This contradicts our assumption that f is onto $(0,1)$. Therefore, $(0,1)$ is not denumerable.

○ $\overline{(0,1)} = c$ (stands for continuum).

It is the only infinite cardinal number to be named other than $\aleph_0$.

DEF: A set S has cardinality c iff S is equivalent to the open interval $(0,1)$; we write $\underline{S} = c$.

EX: The set (a,b) is uncountable

$$f(x) = \frac{x-a}{b-a} \quad \text{is 1-1 and onto } (0,1)$$

○ Thm: The set $\mathbb{R}$ is uncountable and has cardinal number c .

$$f(x) = \tan\left(\pi x - \frac{\pi}{2}\right)$$