

5.3 Countable sets

countable $\rightarrow$ denumerable or finite

Proof that a set is denumerable $\Rightarrow$ Show $A \approx \mathbb{N}$
 $\Rightarrow$ show $f: A \xrightarrow{1-1} \mathbb{N}$

What sets are denumerable? $\Rightarrow \mathbb{N}, \mathbb{E}, \mathbb{Z}, \text{Odd, Whole, what about } \mathbb{N} \times \mathbb{N}$
or $f: \mathbb{N} \xrightarrow{1-1} A$

Can show $\mathbb{N} \times \mathbb{N} \approx \mathbb{N} \Rightarrow f(m, n) = 2^{m-1}(2n-1)$

$\rightarrow$ show f is surjective $\Rightarrow$ (onto) $\Rightarrow$ If $m=1 \Rightarrow$ odd numbers
 $m=2 \Rightarrow$ even "

$\rightarrow$ Show f is 1-1 suppose $f(m, n) = f(r, s)$. Show that
 $m=r$ and $n=s$.

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First, if $f(m, n) = f(r, s)$, then

$$2^{m-1}(2n-1) = 2^{r-1}(2s-1)$$

$$\frac{2^{m-1}}{2^{r-1}}(2n-1) = 2s-1$$

$$2^{m-r}(2n-1) = 2s-1$$

In order for this to be true, both sides must be odd. Thus,
 $m=r$. (otherwise the left side would be even.)

Since $m=r$, then $2n-1 = 2s-1$

$$2n = 2s$$

$$n=s.$$

Thus, $(m, n) = (r, s)$ and f is injective (1-1)

Therefore f is injective and surjective (1-1 and onto)

Therefore, $\mathbb{N} \times \mathbb{N} \approx \mathbb{N}$

What ^{other} sets are denumerable?

How about Q ?

We know that

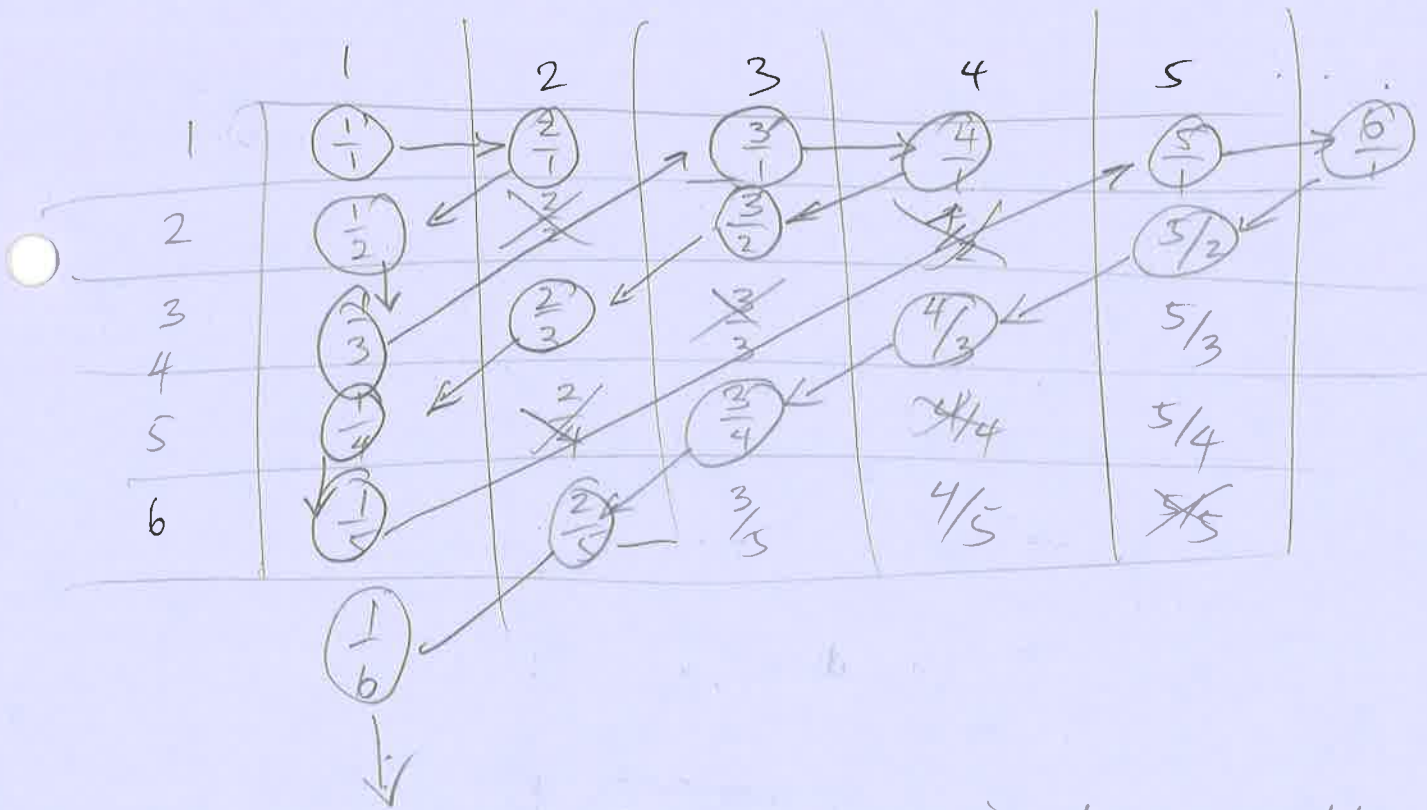
card X0 card C.

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NCQCR

It would seem reasonable that $\overline{Q} = \aleph_0$ or c (or maybe something in between) $\Rightarrow$ continuum hypothesis.

It can be shown that $\bar{Q} = x_0$ and is thus denumerable.



Thm The set of positive rational is denumerable

Thm If A is denumerable, then $A \cup \{x\}$ is denumerable.

proof: If $x \in A$, then $A \cup \{x\} = A$, which is denumerable.

Suppose $x \notin A$. Since $N \times A$, then there exists a 1-1 and onto function $f: N \rightarrow A$ that is onto A .

Define $g: \mathbb{N} \rightarrow A \cup \{x\}$

$$g(n) = \begin{cases} x, & \text{if } n=1 \\ f(n-1), & \text{if } n>1 \end{cases}$$

Since g is 1-1 and onto $A \cup \{x\}$, then

$\mathbb{N} \approx A \cup \{x\} \Rightarrow A \cup \{x\}$ is denumerable.

Do the infinite hotel or
Loew's $\aleph_0$ plex

Thm 5.18 If A is denumerable and B is finite,
then $A \cup B$ is denumerable

Thm 5.19 If A & B are disjoint denumerable sets, then
 $A \cup B$ is denumerable.

Proof: Let $f: \mathbb{N} \xrightarrow{1-1}_{\text{onto}} A$ and $g: \mathbb{N} \xrightarrow{1-1}_{\text{onto}} B$

Define $h(x) = \begin{cases} f(\frac{n+1}{2}), & \text{if } n \text{ is odd} \\ g(\frac{n}{2}), & \text{if } n \text{ is even.} \end{cases}$

Can you show it is a 1-1 correspondence.

Thm 5.20 $\mathbb{Q}$ is denumerable.

Thm 5.21 Let $\{A_i: i=1, 2, \dots, n\}$ be

a finite pairwise disjoint family of denumerable sets. Then $\bigcup_{i=1}^n A_i$ is denumerable.

What about a denumerable set of denumerable sets?

Thm 5.22 Let $\{A_i: i \in \mathbb{N}\}$ be a denumerable pairwise disjoint family of distinct denumerable sets. That is, if $i \in \mathbb{N}$, then A_i is denumerable, and if $i, j \in \mathbb{N}$ and $i \neq j$, then $A_i \cap A_j = \emptyset$. Then $\bigcup_{i \in \mathbb{N}} A_i$ is denumerable.

Thm 5.23 Every subset of a countable set is countable.

Corollary 5.24 A set A is countable iff A is equivalent to some subset of $\mathbb{N}$.

Thm 5.25 If A and B are countable sets, then $A \cup B$ is countable.

Thm 5.26 Let $\mathcal{A}$ be a finite collection of countable sets. Then $\bigcup_{A \in \mathcal{A}} A$ is countable.

Lemma 5.27 Let $\{A_i: i \in \mathbb{N}\}$ be a denumerable family of sets. For each $i \in \mathbb{N}$, let $B_i = A_i - \bigcup_{k=1}^{i-1} A_k$. Then $\{B_i: i \in \mathbb{N}\}$ is a denumerable family of disjoint pairwise sets and $\bigcup_{i \in \mathbb{N}} A_i = \bigcup_{i \in \mathbb{N}} B_i$.

Thm 5.28 Let $\{A_i : i \in \mathbb{N}\}$ be a denumerable family of denumerable sets. Then $\bigcup_{i \in \mathbb{N}} A_i$ is denumerable.

Thm 5.29 Let $\mathcal{A}$ be a countable collection of countable sets. Then $\bigcup \mathcal{A}$ is countable.

Ex: You can also prove $\mathbb{Q}^+$ is denumerable using 5.29.

If we write $\mathbb{Q}^+ = \bigcup_{n \in \mathbb{N}} A_n$, where $A_n = \{\frac{a}{n} : a \in \mathbb{N}\}$

For each n , A_n is denumerable. By thm 5.29, $\mathbb{Q}^+$ is den.

Ex: Each set N_k , $k \in \mathbb{N}$ is a finite set and therefore countable. Their union, $\bigcup_{k \in \mathbb{N}} N_k = \mathbb{N}$ is also countable.

Ex: For each $n \in \mathbb{N}$, $B_n = \{-n, n\}$. The countable union of these countable sets is $\mathbb{Z} - \{0\}$ and is countable!

Ex: Computer programs: consist of lengths.

They are length 1, 2, 3, ..., ...

The $\bigcup$ of all programs is countable only a countable number of them could ever be written in a given language.

However, in 5.1, we saw that the set of all functions from N to $\{0,1\}$ is equivalent to $P(N)$. We shall see that $P(N)$ is uncountable.

This means that there are functions from N to $\{0,1\}$ for which no computer program can compute them. Put another way, there are not enough solutions (programs) for all possible problems (functions).