

5.4 The ordering of the Cardinal numbers

The precise definition is beyond this course. It is sufficient to say that the cardinal number $\bar{A}$ of a set A is an object associated with all sets equivalent to A and no other set. You may also see cardinality notated $|A|$.

DEF: $\bar{A} = \bar{B}$ iff $A \approx B$; otherwise $\bar{A} \neq \bar{B}$.
 $\bar{A} < \bar{B}$ iff there exists $f: A \xrightarrow{1-1} B$
 $\bar{A} < \bar{B}$ iff $\bar{A} \leq \bar{B}$ and $\bar{A} \neq \bar{B}$

Thm 5.30 for sets A, B, C

- (a) (Reflexive) $\bar{A} \leq \bar{A}$
- (b) (Transitivity of $=$) If $\bar{A} = \bar{B}$ and $\bar{B} = \bar{C}$, then $\bar{A} = \bar{C}$
- (c) (Transitivity of $\leq$) If $\bar{A} \leq \bar{B}$ and $\bar{B} \leq \bar{C}$, then $\bar{A} \leq \bar{C}$.
- (d) $\bar{A} \leq \bar{B}$ iff $\bar{A} < \bar{B}$ or $\bar{A} = \bar{B}$
- (e) If $A \subseteq B$, then $\bar{A} \leq \bar{B}$
- (f) $\bar{A} \leq \bar{B}$ iff there is a subset W of B such that $\bar{W} = \bar{A}$.

Ex $K < \aleph_0$ for every finite cardinal number K . By thm 5.30e $N_K \subseteq N$, we have $K \leq \aleph_0$. Since $N_K \neq N$, then $K < \aleph_0$

Ex: Prove $\aleph_0 < c$. First, since $N \subseteq R$, then $\aleph_0 \leq c$. Second, because $N \neq R$, then $\aleph_0 < c$.

Review

If A is a set with n elements, then $P(A)$ is the power set of A and is defined as the set of all subsets of A .

Note for any subset, the set is included in the subset or it is not. Thus, the power set has

$$\underbrace{\underbrace{2}_{\substack{\text{in or out} \\ \text{included or not included}}} \underbrace{2}_{\substack{\text{in or out} \\ \text{included or not included}}} \underbrace{2}_{\substack{\text{in or out} \\ \text{included or not included}}} \underbrace{2}_{\substack{\text{in or out} \\ \text{included or not included}}} \dots \underbrace{2}_{\substack{\text{in or out} \\ \text{included or not included}}}}_{n \text{ times}} = 2^n$$

$$\text{Thus } \overline{\overline{P(A)}} = 2^n \text{ iff } \overline{A} = n$$

Note that $A \not\approx P(A)$ by nature that their cardinal numbers are different. What if A were denumerable, or uncountable?

Thm 5.31 Important! (Cantor's thm)

For every set A , $\overline{A} < \overline{\overline{P(A)}}$.

Proof: To show $\overline{A} < \overline{\overline{P(A)}}$, we must show

$$(i) \overline{A} \leq \overline{\overline{P(A)}}$$

$$(ii) \overline{A} \neq \overline{\overline{P(A)}}$$

Part (i) follows from $F: A \rightarrow P(A)$ defined by $F(x) = \{x\}$ is 1-1. Thus $\overline{A} \leq \overline{\overline{P(A)}}$ by definition (Can use 5.30e since $A \subseteq P(A)$).

Part 2: Suppose $\overline{A} = \overline{\overline{P(A)}}$; that is, assume $A \approx P(A)$. Then there exists a $g: A \xrightarrow[\text{onto}]{1-1} P(A)$.

$$\text{Let } B = \{y \in A : y \notin g(y)\}$$

Since $B \subseteq A$, $B \in P(A)$, and since g is onto $P(A)$, $B = g(z)$ for some $z \in A$. Now, either $z \in B$ or $z \notin B$.

If $z \in B$, then $z \notin g(z) = B$ (def of B), which is a contradiction.

If $z \notin B$, then $z \in g(z) = B$ (def of B), another contradiction.

We conclude that $A \not\approx P(A)$ and hence $\bar{A} < \overline{P(A)}$

Cantor's thm has some interesting consequences:

1) There are infinitely many infinite cardinalities

2) $\aleph_0 = \bar{N}$

By Cantor's thm $\aleph_0 < \overline{\overline{P(N)}}$

Further,

$$\aleph_0 < \overline{\overline{P(N)}} < \overline{\overline{P(P(N))}} < \overline{\overline{P(P(P(N)))}} < \dots$$

Where is c ?

3) There is no largest cardinal number.

In Sec 5.1, we learned that the set of all functions from N to $\{0,1\}$ is equivalent to $P(N)$

Since $\bar{N} < \overline{\overline{P(N)}}$, we know there are ^{an} uncountably infinite number of functions from N to $\{0,1\}$.

Since a function from N to $\{0,1\}$ is a sequence of 0's and 1's, Cantor's thm can be used to prove ~~Bx~~ $B \times B$ of Sec 5.3.

It appears to be obvious that if B has at least as many elements as A ($\bar{A} \leq \bar{B}$) and A has at least as many elements as B ($\bar{B} \leq \bar{A}$), then $A \approx B$ are equivalent. The proof is not obvious.

Thm 5.32 Cantor-Schröder-Bernstein Thm

If $\bar{A} \leq \bar{B}$ and $\bar{B} \leq \bar{A}$, then $\bar{A} = \bar{B}$

Proof: see book.

we can prove equivalence using C-S-B thm when it may be difficult to find a 1-1 correspond.

Ex: show $(0,1) \approx [0,1]$. First note $(0,1) \subseteq [0,1]$, so by thm 5.30e, $\overline{(0,1)} \subseteq \overline{[0,1]}$. Likewise, since $[0,1] \subseteq (-1,2)$, then $\overline{[0,1]} \subseteq \overline{(-1,2)}$. But we know that $(0,1) \approx (-1,2)$ (or any (a,b)). Thus, $\overline{(0,1)} \subseteq \overline{[0,1]}$ and $\overline{[0,1]} \subseteq \overline{(0,1)}$, which means $[0,1] \approx (0,1)$ by C-S-B.

Ex: explore relation of $(0,1)$ to $P(N)$

Recall any number in the interval $(0,1)$ can be represented in base 2 notation as $0.b_1 b_2 b_3 \dots$, where each b_i is either 0 or 1.

If we exclude repeating 1's, then the representation is unique. (Note computers represent fractions this way!). Thus, define the function $f: (0,1) \rightarrow P(N)$ for each x as

$$f(x) = \{n \in N, b_n = 1 \text{ in the binary representation of } x\}$$

The uniqueness of binary representations assures the function is defined and is 1-1. Note that f is not onto $P(N)$ since inductive sets such as $\{5, 6, 7, 8, 9, \dots\}$ are not the images of any $x \in (0,1)$. Since f is 1-1, $\overline{(0,1)} \subseteq P(N)$.

Next, define

$$g: P(N) \rightarrow (0,1) \text{ by } g(A) = 0.a_1 a_2 a_3 \dots$$

$$\text{where } a_n = \begin{cases} 2, & \text{if } n \in A \\ 5, & \text{if } n \notin A \end{cases}$$

for any set $A \subseteq N$, $g(A)$ is a real number in $(0,1)$ with decimal expansion consisting of 2's and 5's (Any pair of #s will do)

The function g is 1-1 but definitely not onto.

$$\text{Therefore } \overline{P(N)} \leq \overline{(0,1)}$$

$$\text{By C-S-B, } \overline{P(N)} = \overline{(0,1)} = C$$

So the sequence

$$\overline{N} \leq \overline{P(N)} < \overline{P(P(N))} < \overline{P(P(P(N)))} < \dots$$

$$\aleph_0 < C < \dots$$

Thm: for sets A, B, C

$$(a) \text{ If } \overline{A} \leq \overline{B}, \text{ then } \overline{B} \neq \overline{A}$$

$$(b) \text{ If } \overline{A} \leq \overline{B} \text{ and } \overline{B} < \overline{C}, \text{ then } \overline{A} < \overline{C}$$

$$(c) \text{ If } \overline{A} < \overline{B} \text{ and } \overline{B} \leq \overline{C}, \text{ then } \overline{A} < \overline{C}$$

$$(d) \text{ If } \overline{A} < \overline{B} \text{ and } \overline{B} < \overline{C}, \text{ then } \overline{A} < \overline{C}$$

What about "If $\overline{B} \neq \overline{A}$, then $\overline{A} \leq \overline{B}$?"
Does that work?