

5.5 Comparability Theorem

Thm 5.34 Comparability thm

If A and B are sets, then $\bar{A} < \bar{B}$, $\bar{A} = \bar{B}$, or $\bar{B} < \bar{A}$

Note: It is impossible to prove thm 5.3.4 from axioms and other thms of Zermelo-Frankel set theory (see sec 2.1)

We also cannot show that (In ZF set theory) that comparability is false. It is undecidable.

At this point we can either assume it is true (or something else that can prove it true) (or something that can show it is false)

It has been standard practice to assume ^{CT} it is true by assuming the following axiom:

The Axiom of Choice

If $\mathcal{A}$ is any collection of nonempty sets, then there exists a function F (called a choice function) from $\mathcal{A}$ to $\bigcup_{A \in \mathcal{A}} A$ such that $\forall A \in \mathcal{A}, F(A) \in A$.

Note that if the collection is finite, then the Axiom isn't needed and regular theory works. It is the infinite collections for which it is valuable. (and not obvious)

Here's an example:

A shoe store has a stockroom with an infinite number of pairs of shoes and an infinite number of pairs of socks. A customer asks to see one shoe of each pair and one sock of each pair. The clerk does not need the Axiom of Choice to select a shoe from each pair. His choice might be "pick the right shoe". However, since the socks of any pair in stock is indistinguishable, and since there are an infinite number of pairs of socks, the clerk must employ the Axiom of Choice to show the customer one sock from each pair.

Ex: Let $\mathcal{A} = \{A: A \subseteq \mathbb{R} \text{ and } A \neq \emptyset\}$.

- If we select one element from each set A in $\mathcal{A}$, then we will need to use the Axiom of Choice.

However, if $\mathcal{B} = \{A: A \subseteq \mathbb{R} \text{ and } A \neq \emptyset, \text{ and } A \text{ is finite}\}$, then we do not need the axiom of choice to select one element from each set in $\mathcal{B}$. In this case, we could say "for each $B \in \mathcal{B}$, choose the greatest element of B . Since B is finite, such an element exists for each $B \in \mathcal{B}$."

Thm 5.22 restated

Let $\{A_i : i \in \mathbb{N}\}$ be a denumerable pairwise disjoint family of distinct denumerable sets, then $\bigcup_{i \in \mathbb{N}} A_i$ is denumerable.

(uses axiom of choice)

Proof: Since each A_i is denumerable, $\exists f_i : \mathbb{N} \rightarrow A_i$, $i \in \mathbb{N}$ each of which is 1-1 and onto A_i .

Define $h : \mathbb{N} \times \mathbb{N} \rightarrow \bigcup_{i \in \mathbb{N}} A_i$ by

$$h(m, n) = f_m(n)$$

$(1, 1)$	$(1, 2)$	$(1, 3)$	$\dots$	A_1	$f_1(1)$	$f_1(2)$	$f_1(3)$	$\dots$
$(2, 1)$	$(2, 2)$	$(2, 3)$	$\dots$	A_2	$f_2(1)$	$f_2(2)$	$f_2(3)$	$\dots$
$(3, 1)$	$(3, 2)$	$(3, 3)$	$\dots$	A_3	$f_3(1)$	$f_3(2)$	$f_3(3)$	$\dots$
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$	

$\underbrace{\hspace{15em}}_{\mathbb{N} \times \mathbb{N}} \qquad \underbrace{\hspace{15em}}_{\bigcup_{i \in \mathbb{N}} A_i}$

Since all the functions f_i are bijections, then it is an easy exercise to show h is a bijection. Therefore,

$\mathbb{N} \times \mathbb{N} \approx \bigcup_{i \in \mathbb{N}} A_i$. Since $\mathbb{N} \times \mathbb{N} \approx \mathbb{N}$, then $\bigcup_{i \in \mathbb{N}} A_i$ is denumerable.

Why was the axiom of choice needed?

For each $i \in \mathbb{N}$, A_i is denumerable, so there were many bijections from A_i to $\mathbb{N}$. In the proof, one was selected from the set of all bijections from A_i to $\mathbb{N}$.

We did this an infinite number of times, once for each $i \in \mathbb{N}$.

Our collection consisted of sets of bijections, and we needed one bijection from each set. There is no way to select

f_i without the axiom of choice.

Read pg 240 Paragraph 2 & 3.

Thm 5.35 If there exists a function from A onto B ,
then $\bar{B} \leq \bar{A}$

Thm 5.36 Every infinite set A has a denumerable subset.

(Note that this can be used to show every infinite set is equivalent to one of its proper subsets.)

It also says that $\aleph_0$ is the smallest infinite cardinal number.

Corollary A non-empty set A is countable iff there exists a function $f: \mathbb{N} \xrightarrow{\text{onto}} A$.

(Continuum Hypothesis)

We know that

$$\underbrace{\bar{\mathbb{N}}}_{\aleph_0} < \overline{P(\mathbb{N})} < \overline{P(P(\mathbb{N}))} < \overline{P(P(P(\mathbb{N})))} < \dots$$
$$\aleph_0 < c$$

Is there a cardinal number $\bar{x}$ such that

$$\aleph_0 < \bar{x} < c \quad ?$$

Famous problem! Proved by Gödel & Cohen that it can neither be proved or disproved using Zermelo-Frankel set theory!
Undecidable!