Math 311

Numerical Methods

2.1: Topic

Probability, Random Variables, and Distributions

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1 How does the square root button on a calculator work?

1.1 8th Grade Method

When I was a kid in the 8th grade, we were first introduced to "how do we compute" a square root. In short, this is what my class (and teacher) did:

- First, make a guess. (e.g. use 2.5 as the square root of 5). Then:
  1. Next, multiply 2.5×2.5 by hand (no calculators allowed).
  2. Based on the answer from the last step, either lower or raise your estimate.
     - If the square is less than 5, then increase the guess a little bit.
     - If the square is greater than 5, then decrease the guess a little bit.
  3. Then repeat step 1.
- The last two steps are then repeated until the answer gets sufficiently close.
- How to update the estimate was never taught to us! It was a guess!
- It was tedious, since it required multiplying longer and longer length numbers.
- The more digits our estimate, the longer to multiply out by hand. Thus, the problem gets longer and longer.

1.2 A better Square Root Algorithm

Since eighth grade, I have found algorithms that work way better than this. The oldest was the Babylonian Method (used thousands of years ago):

Babylonian Method

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$$

- Bisection Method can do it. So can Newton's method.
- I sometimes play around with it.
- So, while playing with my calculator, I came up with what I thought was a new method.
- Here's what I did:
  - Suppose I wanted to find the square root of 50.
  - If 50 is divided by the exact $\sqrt{50}$, it gives the true answer.
  - What if we divided by the closest integer to the answer?
  - It should be "close" to the actual answer.

- Since 7 is the square root of 49, then I divided 50 by 7 and considered that maybe this was close enough.
- The answer is
$$\frac{50}{7} = \texttt{7.142857142857}$$
- This answer was a pretty good estimate itself, but not any better than 7.
- Since the real square root is in between 7 and this number, I thought, why not try and <u>average</u> the two estimates, and use that as the new guess?
- Doing so gives
$$\frac{1}{2}\Big(\texttt{7+7.142857142857}\Big) = \texttt{7.07142857145}$$
- That was a pretty good estimate. Gets you 50.0051 upon squaring. Pretty good.
- I thought, it worked once, let's do it again! Do:
$$\frac{1}{2}\left(\texttt{7.07142857145} + \frac{\texttt{50}}{\texttt{7.07142857145}}\right) = \texttt{7.07106782105}.$$
- Wow! This one is better! I get 50.0000001299 upon squaring it!
- Let's do it again:
$$\frac{1}{2}\left(\texttt{7.07106782105} + \frac{\texttt{50}}{\texttt{7.07106782105}}\right) = \texttt{7.07106781185}.$$

- Squaring that answer gives `49.9999999998`! I was astounded!
- Three iterations to get 11 digits of accuracy?! That's fast. I excitedly thought that I had come up with a really clever way to find square roots.
- Sadly, though, I had only reproduced **Newton's method**! Here's how the method was justified in my mind:
- If we want to find $x = \sqrt{r}$, we can do it iteratively, where

$$x^2 = r \Longrightarrow x \cdot x = r \Longrightarrow x = \frac{r}{x}$$

- We have two estimates, x and r/x. Our method is simply average them, then rinse and repeat!

$$x_{new} = \frac{1}{2}\left(x + \frac{r}{x}\right).$$

- The convergence of this method is really fast!
- It only requires at most a half dozen to converge to 13 digits.
- Applying Newton's method to $f(x) = x^2 - r$ produces the same method.
- Why does it work so fast?
- We can use the Fixed Point Theorem to see why.

Theorem. *Fixed Point Theorem states:*
If p is the fixed point of $g(x)$ (meaning $g(p) = p$), then $g'(p) = 0$.

The fixed point for this problem is $p = \sqrt{r}$ and the function $g(x)$ is $\frac{1}{2}\left(x + \frac{r}{x}\right)$. So the derivative is

$$g'(x) = \frac{1}{2} - \frac{r}{2x^2}$$

Evaluating at p yields

$$g'(p) = \frac{1}{2} - \frac{r}{2(\sqrt{r})^2} = \frac{1}{2} - \frac{1}{2} = 0.$$

So it follows that the Fixed Pt theorem guarantees that this method has at leaast quadratic convergence!

2 Cubic Roots

- I thought, why not try this method with Cubic Roots? So, I tried the same idea: If we want to find $x = \sqrt[3]{r}$, we can do it iteratively, where

$$x^3 = r \Longrightarrow x \cdot x^2 = r \Longrightarrow x = \frac{r}{x^2}$$

- We have two estimates, x and r/x^2. Our method is simply average them, then rinse and repeat!

$$x_{new} = \frac{1}{2}\left(x + \frac{r}{x^2}\right).$$

- The convergence of this method is unfortunately slow!
- It is quite slow. What can we do to accelerate the convergence?
- Let's use the theorem above to accelerate the convergence!
- What about a weighted average instead for $g(x)$? Let $0 < \alpha < 1$ and make the update be:

$$x_{new} = \alpha x + (1 - \alpha)\frac{r}{x^2} = g(x)$$

- To accelerate, we want to find the best choice for α that forces $g'(p) = 0$.

- Since $p = \sqrt[3]{r}$, then

$$g'(x) = \alpha - 2(1-\alpha)\frac{r}{x^3}$$
$$g'(p) = g'(\sqrt[3]{r}) = \alpha - 2(1-\alpha)\frac{r}{(\sqrt[3]{r})^3} = \alpha - 2(1-\alpha) = 3\alpha - 2$$

- It follows that $\alpha = \dfrac{2}{3}$ is the best answer. Therefore, our method should be

Cool Fast Cube Root

$$x_{new} = \frac{2}{3}x + \frac{r}{3x^2}$$

- This new method converges quadratically AND is equivalent to Newton's method applied to $f(x) = x^3 - r$.

3 General Roots

We can repeat the same steps as the preceding methods and produce a method that is quadratically convergent for finding the m^{th} root of a number. Details are in the handout "Thoughts on Roots".
The method is

Cool Fast Cube Root

$$x_{new} = \frac{m-1}{m}x + \frac{r}{mx^{m-1}} = \frac{1}{m}\left((m-1)x + \frac{r}{x^{m-1}}\right)$$

- This method converges quadratically! It is also equivalent to Newton's method applied to $f(x) = x^m - r$.
- While we have not really created anything new, we have given a different motivation to how methods can be developed.