Example for each method in Chap 3

$f(x) = \frac{1}{x}$

x	$\frac{2}{3}$	1	3	4
y	$\frac{3}{2}$	1	$\frac{1}{3}$	$\frac{1}{4}$

we are fitting $f(x) = \frac{1}{x}$ for 4 values of x

Lagrange Polynomial

$$L_0(x) = \frac{(x-1)(x-3)(x-4)}{\left(\frac{2}{3}-1\right)\left(\frac{2}{3}-3\right)\left(\frac{2}{3}-4\right)} = -\left(\frac{27x^3-216x^2+513x-324}{70}\right)$$

$$L_1(x) = \frac{\left(x-\frac{2}{3}\right)(x-3)(x-4)}{\left(1-\frac{2}{3}\right)(1-3)(1-4)} = \frac{3x^3-23x^2+50x-24}{6}$$

$$L_2(x) = \frac{\left(x-\frac{2}{3}\right)(x-1)(x-4)}{\left(3-\frac{2}{3}\right)(3-1)(3-4)} = -\left(\frac{3x^3-17x^2+22x-8}{14}\right)$$

$$L_3(x) = \frac{\left(x-\frac{2}{3}\right)(x-1)(x-3)}{\left(4-\frac{2}{3}\right)(4-1)(4-3)} = \frac{3x^3-14x^2+17x-6}{30}$$

$$P(x) = \frac{3}{2}L_0(x) + 1\cdot L_1(x) + \frac{1}{3}L_2(x) + \frac{1}{4}L_3(x)$$

$$= \frac{3}{2}\left(\frac{-27x^3+216x^2-513x+324}{70}\right) + \left(\frac{3x^3-23x^2+50x-24}{6}\right)$$

$$- \frac{1}{3}\left(\frac{3x^3-17x^2+22x-8}{14}\right) + \frac{1}{4}\left(\frac{3x^3-14x^2+17x-6}{30}\right)$$

= simplifies to Vandermonde answer

Vandermonde Matrix

$$V = \begin{bmatrix} 1 & x & x^2 & x^3 \end{bmatrix} = \begin{bmatrix} 1 & 2/3 & 4/9 & 8/27 \\ 1 & 1 & 1 & 1 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix} \qquad b = \begin{bmatrix} 3/2 \\ 1 \\ 1/3 \\ 1/4 \end{bmatrix}$$

The solution is $P(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$

using a calculator

$$X = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = V^{-1} b = \frac{1}{210} \begin{bmatrix} 972 & -840 & 120 & -42 \\ -1539 & 1750 & -330 & 119 \\ 648 & -805 & 255 & -98 \\ -81 & 105 & -45 & 21 \end{bmatrix} \begin{bmatrix} 3/2 \\ 1 \\ 1/3 \\ 1/4 \end{bmatrix}$$

$$= \frac{1}{210(12)} \begin{bmatrix} 972 & -840 & 120 & -42 \\ -1539 & 1750 & -330 & 119 \\ 648 & -805 & +255 & -98 \\ -81 & 105 & -45 & 21 \end{bmatrix} \begin{bmatrix} 18 \\ 12 \\ 4 \\ 3 \end{bmatrix}$$

$$= \frac{1}{70 \cdot 3 \cdot 12} \begin{bmatrix} 7770 \\ -7665 \\ 2730 \\ -315 \end{bmatrix} = \frac{1}{24} \left(74 - 73x + 26x^2 - 3x^3 \right)$$

Newton's IDD (For nonequal spacing)

x	y			
$\frac{2}{3}$	$\frac{3}{2}$	$-\frac{3}{2}$	$\frac{1}{2}$	$-\frac{1}{8}$
1	1	$-\frac{1}{3}$	$\frac{1}{12}$	
3	$\frac{1}{3}$	$-\frac{1}{12}$		
4	$\frac{1}{4}$			

$$P(x) = \frac{3}{2} - \frac{3}{2}\left(x-\frac{2}{3}\right) + \frac{1}{2}\left(x-\frac{2}{3}\right)(x-1) - \frac{1}{8}\left(x-\frac{2}{3}\right)(x-1)(x-3)$$

$$= -\frac{1}{24}\left(3x^3 - 26x^2 + 73x - 74\right)$$

Simplifies to Vandermonde

Hermite Polynomial

$$f(x) = \frac{1}{x}$$

$$f'(x) = -\frac{1}{x^2}$$

x	y	y_p
2/3	3/2	-9/4
1	1	-1
3	1/3	-1/9
4	1/4	-1/16

get inserted where □'s are

Hermite Table

x	y							
2/3	(3/2)							
2/3	3/2	[-9/4]						
1	1	-3/2	(9/4)					
1	1	[-1]	3/2	(-9/4)				
3	1/3	-1/3	1/3	-1/2	(3/4)			
3	1/3	[-1/9]	1/9	-1/9	1/6	(-1/4)		
4	1/4	-1/12	1/36	-1/36	1/31	-1/24	(1/16)	
4	1/4	[-1/16]	1/48	-1/144	1/144	-1/144	1/96	(-1/64)

Next, take circled numbers and create polynomial like Newton's Int. div. diff,

$$H_7(x) = \frac{3}{2} - \frac{9}{4}\left(x-\frac{2}{3}\right)$$

$$+\frac{9}{4}\left(x-\frac{2}{3}\right)^2$$

$$-\frac{9}{4}\left(x-\frac{2}{3}\right)^2(x-1)$$

$$+\frac{3}{4}\left(x-\frac{2}{3}\right)^2(x-1)^2$$

$$-\frac{1}{4}\left(x-\frac{2}{3}\right)^2(x-1)^2(x-3)$$

$$+\frac{1}{16}\left(x-\frac{2}{3}\right)^2(x-1)^2(x-3)^2$$

$$-\frac{1}{64}\left(x-\frac{2}{3}\right)^2(x-1)^2(x-3)^2(x-4)$$

spline (A)

Cubic Spline

x	y	y_P
2/3	3/2	-9/4
1	1	-1
3	1/3	-1/9
4	1/4	-1/16

Natural Spline

$$S = \begin{cases} \frac{3}{2} - \frac{229}{144}\left(x-\frac{2}{3}\right) + \frac{13}{16}\left(x-\frac{2}{3}\right)^3, & \frac{2}{3} \le x < 1 \\ 1 - \frac{95}{72}(x-1) + \frac{13}{16}(x-1)^2 - \frac{23}{144}(x-1)^3, & 1 \le x < 3 \\ \frac{1}{3} + \frac{1}{72}(x-3) - \frac{7}{48}(x-3)^2 + \frac{7}{144}(x-3)^3, & 3 \le x < 4 \end{cases}$$

Clamped

spline (A, "clamped", -9/4, -1/16)

$$S = \begin{cases} \frac{3}{2} - \frac{9}{4}\left(x-\frac{2}{3}\right) + \frac{2055}{664}\left(x-\frac{2}{3}\right)^2 - \frac{1683}{664}\left(x-\frac{2}{3}\right)^3, & \frac{2}{3} \le x < 1 \\ 1 - \frac{685}{664}(x-1) + \frac{93}{166}(x-1)^2 - \frac{841}{7968}(x-1)^3, & 1 \le x < 3 \\ \frac{1}{3} - \frac{19}{332}(x-3) - \frac{97}{1328}(x-3)^2 + \frac{187}{3984}(x-3)^2, & 3 \le x < 4 \end{cases}$$