Interpolation & Polynomial Approximation

Hermite Interpolation I

Numerical Analysis (9th Edition)

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Outline

1. Osculating & Hermite Polynomials

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Hermite Interpolation: Osculating Polynomials

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Hermite Interpolation: Osculating Polynomials

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- The osculating polynomial approximating a function $f \in C^m[a, b]$ at x_i, for each $i = 0, \ldots, n$, is the polynomial of least degree that has the same values as the function f and all its derivatives of order less than or equal to m_i at each x_i.

Hermite Interpolation: Osculating Polynomials

Osculating Polynomials (Cont'd)

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Osculating Polynomials (Cont'd)

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because the number of conditions to be satisfied is $\sum_{i=0}^{n} m_i + (n+1)$, and a polynomial of degree M has $M+1$ coefficients that can be used to satisfy these conditions.

Hermite Interpolation: Osculating Polynomials

Definition: Osculating Polynomial

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- Suppose that $f \in C^m[a, b]$, where $m = \max_{0 \le i \le n} m_i$.
- The osculating polynomial approximating f is the polynomial $P(x)$ of least degree such that

$$\frac{d^k P(x_i)}{dx^k} = \frac{d^k f(x_i)}{dx^k}$$

for each $i = 0, 1, \ldots, n$ and $k = 0, 1, \ldots, m_i$.

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The osculating polynomial approximating f is the m_0th Taylor polynomial for f at x_0 when $n = 0$ and the nth Lagrange polynomial interpolating f on $x_0, x_1, \ldots, x_n$ when $m_i = 0$ for each i.

Hermite Polynomials

Osculating Polynomials

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Hermite Polynomials

- The case when $m_i = 1$, for each $i = 0, 1, \ldots, n$, gives the Hermite polynomials.

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Hermite Polynomials

- The case when $m_i = 1$, for each $i = 0, 1, \ldots, n$, gives the Hermite polynomials.
- For a given function f, these polynomials agree with f at $x_0, x_1, \ldots, x_n$.
- In addition, since their first derivatives agree with those of f, they have the same "shape" as the function at $(x_i, f(x_i))$ in the sense that the tangent lines to the polynomial and the function agree.

Outline

1. Osculating & Hermite Polynomials
2. The Precise Form of the Hermite Polynomials
3. Example: Constructing the Hermite Polynomial using Lagrange Polynomials

Precise Form of the Hermite Polynomials

Theorem

If $f \in C^1[a, b]$ and $x_0, \ldots, x_n \in [a, b]$ are distinct, the unique polynomial of least degree agreeing with f and f' at $x_0, \ldots, x_n$ is the Hermite polynomial of degree at most $2n + 1$ given by

$$H_{2n+1}(x) = \sum_{j=0}^{n} f(x_j) H_{n,j}(x) + \sum_{j=0}^{n} f'(x_j) \hat{H}_{n,j}(x)$$

where, for $L_{n,j}(x)$ denoting the jth Lagrange coefficient polynomial of degree n, we have

$$H_{n,j}(x) = [1 - 2(x - x_j) L'_{n,j}(x_j)] L^2_{n,j}(x) \quad \text{and} \quad \hat{H}_{n,j}(x) = (x - x_j) L^2_{n,j}(x)$$

Continued on the next slide . . .

Precise Form of the Hermite Polynomials

$$H_{2n+1}(x) = \sum_{j=0}^{n} f(x_j)H_{n,j}(x) + \sum_{j=0}^{n} f'(x_j)\hat{H}_{n,j}(x)$$

Theorem (Cont'd)

Moreover, if $f \in C^{2n+2}[a, b]$, then

$$f(x) = H_{2n+1}(x) + \frac{(x - x_0)^2 \ldots (x - x_n)^2}{(2n+2)!} f^{(2n+2)}(\xi(x))$$

for some (generally unknown) $\xi(x)$ in the interval (a, b).

Precise Form of the Hermite Polynomials

Proof (1/4)

First recall that

$$L_{n,j}(x_i) = \begin{cases} 0, & \text{if } i \neq j \\ 1, & \text{if } i = j \end{cases}$$

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$$H_{n,j}(x_i) = 0 \qquad \text{and} \qquad \hat{H}_{n,j}(x_i) = 0$$

whereas, for each i,

$$\begin{aligned} H_{n,i}(x_i) &= [1 - 2(x_i - x_i)L'_{n,i}(x_i)] \cdot 1 = 1 \\ \text{and} \quad \hat{H}_{n,i}(x_i) &= (x_i - x_i) \cdot 1^2 = 0 \end{aligned}$$

Precise Form of the Hermite Polynomials

Proof (2/4)

- As a consequence

$$H_{2n+1}(x_i) = \sum_{\substack{j=0 \\ j\neq i}}^{n} f(x_j)\cdot 0 + f(x_i)\cdot 1 + \sum_{j=0}^{n} f'(x_j)\cdot 0 = f(x_i)$$

so H_{2n+1} agrees with f at $x_0, x_1, \ldots, x_n$.

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  so H_{2n+1} agrees with f at $x_0, x_1, \ldots, x_n$.
- To show the agreement of H'_{2n+1} with f' at the nodes, first note that $L_{n,j}(x)$ is a factor of $H'_{n,j}(x)$, so $H'_{n,j}(x_i) = 0$ when $i \neq j$.

Precise Form of the Hermite Polynomials

Proof (3/4)

In addition, when $i = j$ we have $L_{n,i}(x_i) = 1$, so

$$\begin{aligned} H'_{n,i}(x_i) &= -2L'_{n,i}(x_i) \cdot L^2_{n,i}(x_i) + [1 - 2(x_i - x_i)L'_{n,i}(x_i)]2L_{n,i}(x_i)L'_{n,i}(x_i) \\ &= -2L'_{n,i}(x_i) + 2L'_{n,i}(x_i) = 0 \end{aligned}$$

Hence, $H'_{n,j}(x_i) = 0$ for all i and j.

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Hence, $H'_{n,j}(x_i) = 0$ for all i and j.
Finally,

$$\begin{aligned} \hat{H}'_{n,j}(x_i) &= L^2_{n,j}(x_i) + (x_i - x_j)2L_{n,j}(x_i)L'_{n,j}(x_i) \\ &= L_{n,j}(x_i)[L_{n,j}(x_i) + 2(x_i - x_j)L'_{n,j}(x_i)] \end{aligned}$$

so $\hat{H}'_{n,j}(x_i) = 0$ if $i \neq j$ and $\hat{H}'_{n,i}(x_i) = 1$.

Precise Form of the Hermite Polynomials

Proof (4/4)

Combining these facts, we have

$$H'_{2n+1}(x_i) = \sum_{j=0}^{n} f(x_j) \cdot 0 + \sum_{\substack{j=0 \\ j \neq i}}^{n} f'(x_j) \cdot 0 + f'(x_i) \cdot 1 = f'(x_i)$$

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Therefore, H_{2n+1} agrees with f and H'_{2n+1} with f' at $x_0, x_1, \ldots, x_n$.

Outline

Constructing the Hermite Polynomial

Example: Constructing $H_5(x)$

Use the Hermite polynomial that agrees with the data listed in the following table to find an approximation to $f(1.5)$.

k	x_k	$f(x_k)$	$f'(x_k)$
0	1.3	0.6200860	−0.5220232
1	1.6	0.4554022	−0.5698959
2	1.9	0.2818186	−0.5811571

Constructing the Hermite Polynomial $H_5(x)$

Solution (1/5)

We first compute the Lagrange polynomials and their derivatives.

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$$L_{2,0}(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = \frac{50}{9}x^2 - \frac{175}{9}x + \frac{152}{9}$$

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Constructing the Hermite Polynomial $H_5(x)$

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\begin{aligned}
L_{2,0}(x) &= \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = \frac{50}{9}x^2 - \frac{175}{9}x + \frac{152}{9} \\
L'_{2,0}(x) &= \frac{100}{9}x - \frac{175}{9} \\
L_{2,1}(x) &= \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = \frac{-100}{9}x^2 + \frac{320}{9}x - \frac{247}{9}
\end{aligned}
$$

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L'_{2,0}(x) &= \frac{100}{9}x - \frac{175}{9} \\
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L'_{2,1}(x) &= \frac{-200}{9}x + \frac{320}{9}
\end{aligned}
$$

Constructing the Hermite Polynomial $H_5(x)$

Solution (2/5)

and, finally

$$L_{2,2} = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} = \frac{50}{9}x^2 - \frac{145}{9}x + \frac{104}{9}$$

Constructing the Hermite Polynomial $H_5(x)$

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$$L'_{2,2}(x) = \frac{100}{9}x - \frac{145}{9}$$

Constructing the Hermite Polynomial $H_5(x)$

Solution (3/5)

The polynomials $H_{2,j}(x)$ and $\hat{H}_{2,j}(x)$ are then

$$H_{2,0}(x) = [1 - 2(x - 1.3)(-5)]\left(\frac{50}{9}x^2 - \frac{175}{9}x + \frac{152}{9}\right)^2$$

Constructing the Hermite Polynomial $H_5(x)$

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The polynomials $H_{2,j}(x)$ and $\hat{H}_{2,j}(x)$ are then

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\begin{aligned}
H_{2,0}(x) &= [1 - 2(x-1.3)(-5)]\left(\frac{50}{9}x^2 - \frac{175}{9}x + \frac{152}{9}\right)^2 \\
&= (10x - 12)\left(\frac{50}{9}x^2 - \frac{175}{9}x + \frac{152}{9}\right)^2
\end{aligned}
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\begin{aligned}
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&= (10x-12)\left(\frac{50}{9}x^2-\frac{175}{9}x+\frac{152}{9}\right)^2 \\
H_{2,1}(x) &= 1\cdot\left(\frac{-100}{9}x^2+\frac{320}{9}x-\frac{247}{9}\right)^2 \\
H_{2,2}(x) &= 10(2-x)\left(\frac{50}{9}x^2-\frac{145}{9}x+\frac{104}{9}\right)^2
\end{aligned}
$$

Constructing the Hermite Polynomial $H_5(x)$

Solution (4/5)

$$\hat{H}_{2,0}(x) = (x - 1.3)\left(\frac{50}{9}x^2 - \frac{175}{9}x + \frac{152}{9}\right)^2$$

Constructing the Hermite Polynomial $H_5(x)$

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$$\hat{H}_{2,0}(x) = (x-1.3)\left(\frac{50}{9}x^2 - \frac{175}{9}x + \frac{152}{9}\right)^2$$

$$\hat{H}_{2,1}(x) = (x-1.6)\left(\frac{-100}{9}x^2 + \frac{320}{9}x - \frac{247}{9}\right)^2$$

Constructing the Hermite Polynomial $H_5(x)$

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\hat{H}_{2,0}(x) &= (x-1.3)\left(\frac{50}{9}x^2 - \frac{175}{9}x + \frac{152}{9}\right)^2 \\
\hat{H}_{2,1}(x) &= (x-1.6)\left(\frac{-100}{9}x^2 + \frac{320}{9}x - \frac{247}{9}\right)^2 \\
\hat{H}_{2,2}(x) &= (x-1.9)\left(\frac{50}{9}x^2 - \frac{145}{9}x + \frac{104}{9}\right)^2
\end{aligned}$$

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$$\begin{aligned} H_5(x) = {} & 0.6200860 H_{2,0}(x) + 0.4554022 H_{2,1}(x) + 0.2818186 H_{2,2}(x) \\ & - 0.5220232 \hat{H}_{2,0}(x) - 0.5698959 \hat{H}_{2,1}(x) \\ & - 0.5811571 \hat{H}_{2,2}(x) \end{aligned}$$

so that

$$\begin{aligned} H_5(1.5) = {} & 0.6200860 \left(\frac{4}{27}\right) + 0.4554022 \left(\frac{64}{81}\right) + 0.2818186 \left(\frac{5}{81}\right) \\ & - 0.5220232 \left(\frac{4}{405}\right) - 0.5698959 \left(\frac{-32}{405}\right) \\ & - 0.5811571 \left(\frac{-2}{405}\right) = 0.5118277 \end{aligned}$$

a result that is accurate to the places listed.

Constructing the Hermite Polynomial

Observation

Although the theorem provides a complete description of the Hermite polynomials, it is clear from this example that the need to determine and evaluate the Lagrange polynomials and their derivatives makes the procedure tedious even for small values of *n*.

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Remedy

We will turn to an alternative method for generating Hermite approximations that has as its basis the Newton interpolatory divided-difference formula at $x_0, x_1, \ldots, x_n$, that is,

$$P_n(x) = f[x_0] + \sum_{k=1}^{n} f[x_0, x_1, \ldots, x_k](x - x_0) \cdots (x - x_{k-1})$$

Questions?