Interpolation & Polynomial Approximation

Hermite Interpolation II

Numerical Analysis (9th Edition)

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Outline

1. Hermite Polynomials Using Divided Differences

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2. Example: Computing $H_5(1.5)$ Using Divided Differences

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Hermite Polynomials & Divided Differences

Introduction

There is an alternative method for generating Hermite approximations that has as its basis the Newton interpolatory divided-difference formula at $x_0, x_1, \ldots, x_n$, that is,

$$P_n(x) = f[x_0] + \sum_{k=1}^{n} f[x_0, x_1, \ldots, x_k](x - x_0) \cdots (x - x_{k-1}).$$

Hermite Polynomials & Divided Differences

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The alternative method uses the connection between the nth divided difference and the nth derivative of f. See Theorem

Hermite Polynomials & Divided Differences

Construction

- Suppose that the distinct numbers $x_0, x_1, \ldots, x_n$ are given together with the values of f and f' at these numbers. Define a new sequence $z_0, z_1, \ldots, z_{2n+1}$ by

$$z_{2i} = z_{2i+1} = x_i, \quad \text{for each } i = 0, 1, \ldots, n,$$

and construct the divided difference table ▸ See Original Table in a form that uses z_0, z_1, . . ., z_{2n+1}.

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  and construct the divided difference table See Original Table in a form that uses z_0, z_1, ..., z_{2n+1}.

- Since $z_{2i} = z_{2i+1} = x_i$ for each i, we cannot define $f[z_{2i}, z_{2i+1}]$ by the divided difference formula. However, we will assume, based on the divided-difference theorem See Theorem that the reasonable substitution in this situation is $f[z_{2i}, z_{2i+1}] = f'(z_{2i}) = f'(x_i)$.

Hermite Polynomials & Divided Differences

Construction (Cont'd)

- Under this assumption, we can use the entries

$$f'(x_0), f'(x_1), \ldots, f'(x_n)$$

in place of the undefined first divided differences

$$f[z_0, z_1], f[z_2, z_3], \ldots, f[z_{2n}, z_{2n+1}]$$

Hermite Polynomials & Divided Differences

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- The following table shows the entries that are used for the first three divided-difference columns when determining the Hermite polynomial $H_5(x)$ for x_0, x_1, and x_2.

Hermite Polynomials & Divided Differences

z	$f(z)$	First divided differences	Second divided differences
$z_0 = x_0$	$f[z_0] = f(x_0)$		
		$f[z_0, z_1] = f'(x_0)$	
$z_1 = x_0$	$f[z_1] = f(x_0)$		$f[z_0, z_1, z_2] = \dfrac{f[z_1, z_2] - f[z_0, z_1]}{z_2 - z_0}$
		$f[z_1, z_2] = \dfrac{f[z_2] - f[z_1]}{z_2 - z_1}$	
$z_2 = x_1$	$f[z_2] = f(x_1)$		$f[z_1, z_2, z_3] = \dfrac{f[z_2, z_3] - f[z_1, z_2]}{z_3 - z_1}$
		$f[z_2, z_3] = f'(x_1)$	
$z_3 = x_1$	$f[z_3] = f(x_1)$		$f[z_2, z_3, z_4] = \dfrac{f[z_3, z_4] - f[z_2, z_3]}{z_4 - z_2}$
		$f[z_3, z_4] = \dfrac{f[z_4] - f[z_3]}{z_4 - z_3}$	
$z_4 = x_2$	$f[z_4] = f(x_2)$		$f[z_3, z_4, z_5] = \dfrac{f[z_4, z_5] - f[z_3, z_4]}{z_5 - z_3}$
		$f[z_4, z_5] = f'(x_2)$	
$z_5 = x_2$	$f[z_5] = f(x_2)$		

Hermite Polynomials & Divided Differences

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Hermite Polynomial: Divided-Difference Form

The Hermite polynomial is then given by

$$H_{2n+1}(x) = f[z_0] + \sum_{k=1}^{2n+1} f[z_0, \ldots, z_k](x - z_0)(x - z_1) \cdots (x - z_{k-1})$$

A proof of this fact can be found in [Pow], p. 56.

Outline

1. Hermite Polynomials Using Divided Differences
2. Example: Computing $H_5(1.5)$ Using Divided Differences
3. The Hermite Interpolation Algorithm

Hermite Polynomials & Divided Differences

Example: Computing $H_5(1.5)$ Using Divided Differences

Use the divided difference method to construct the Hermite polynomial that agrees with the data listed in the following table to find an approximation to $f(1.5)$.

k	x_k	$f(x_k)$	$f'(x_k)$
0	1.3	0.6200860	−0.5220232
1	1.6	0.4554022	−0.5698959
2	1.9	0.2818186	−0.5811571

Hermite Polynomials & Divided Differences

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Note: The underlined entries in the first three columns of the following table are the data given from the earlier example. The remaining entries are generated by the standard divided-difference formula.

Hermite Polynomials & Divided Differences

Solution (1/3)

1.3	0.6200860					
		−0.5220232				
1.3	0.6200860		−0.0897427			
		−0.5489460		0.0663657		
1.6	0.4554022		−0.0698330		0.0026663	
		−0.5698959		0.0679655		−0.0027738
1.6	0.4554022		−0.0290537		0.0010020	
		−0.5786120		0.0685667		
1.9	0.2818186		−0.0084837			
		−0.5811571				
1.9	0.2818186					

Hermite Polynomials & Divided Differences

Solution (2/3)

For example, for the second entry in the third column we use the second 1.3 entry in the second column and the first 1.6 entry in that column to obtain

$$\frac{0.4554022 - 0.6200860}{1.6 - 1.3} = -0.5489460.$$

Hermite Polynomials & Divided Differences

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$$\frac{0.4554022 - 0.6200860}{1.6 - 1.3} = -0.5489460.$$

For the first entry in the fourth column we use the first 1.3 entry in the third column and the first 1.6 entry in that column to obtain

$$\frac{-0.5489460 - (-0.5220232)}{1.6 - 1.3} = -0.0897427.$$

Hermite Polynomials & Divided Differences

Solution (3/3)

The value of the Hermite polynomial at 1.5 is

$$
\begin{aligned}
& H_5(1.5) \\
= & f[1.3] + f'(1.3)(1.5-1.3) + f[1.3,1.3,1.6](1.5-1.3)^2 \\
& + f[1.3,1.3,1.6,1.6](1.5-1.3)^2(1.5-1.6) \\
& + f[1.3,1.3,1.6,1.6,1.9](1.5-1.3)^2(1.5-1.6)^2 \\
& + f[1.3,1.3,1.6,1.6,1.9,1.9](1.5-1.3)^2(1.5-1.6)^2(1.5-1.9) \\
= & 0.6200860 + (-0.5220232)(0.2) + (-0.0897427)(0.2)^2 \\
& + 0.0663657(0.2)^2(-0.1) + 0.0026663(0.2)^2(-0.1)^2 \\
& + (-0.0027738)(0.2)^2(-0.1)^2(-0.4) \\
= & 0.5118277
\end{aligned}
$$

Outline

The Hermite Interpolation Algorithm (1/2)

To obtain the coefficients of the Hermite interpolating polynomial $H(x)$ on the $(n+1)$ distinct numbers $x_0, \ldots, x_n$ for the function f:

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INPUT numbers $x_0, x_1, \ldots, x_n$; values $f(x_0), \ldots, f(x_n)$ and $f'(x_0)$, ..., $f'(x_n)$

The Hermite Interpolation Algorithm (1/2)

To obtain the coefficients of the Hermite interpolating polynomial $H(x)$ on the $(n+1)$ distinct numbers $x_0, \ldots, x_n$ for the function f:

INPUT numbers $x_0, x_1, \ldots, x_n$; values $f(x_0), \ldots, f(x_n)$ and $f'(x_0)$, …, $f'(x_n)$

OUTPUT the numbers $Q_{0,0}, Q_{1,1}, \ldots, Q_{2n+1,2n+1}$ where

$$\begin{aligned} H(x) = \; & Q_{0,0} + Q_{1,1}(x-x_0) + Q_{2,2}(x-x_0)^2 \\ & + Q_{3,3}(x-x_0)^2(x-x_1) + Q_{4,4}(x-x_0)^2(x-x_1)^2 + \cdots \\ & + Q_{2n+1,2n+1}(x-x_0)^2(x-x_1)^2 \cdots (x-x_{n-1})^2(x-x_n) \end{aligned}$$

The Hermite Interpolation Algorithm (2/2)

Step 1 For $i = 0, 1, \ldots, n$ do Steps 2 and 3:

Step 2 Set
$$\begin{aligned} z_{2i} &= x_i \\ z_{2i+1} &= x_i \\ Q_{2i,0} &= f(x_i) \\ Q_{2i+1,0} &= f(x_i) \\ Q_{2i+1,1} &= f'(x_i) \end{aligned}$$

Step 3 If $i \neq 0$ then set
$$Q_{2i,1} = \frac{Q_{2i,0} - Q_{2i-1,0}}{z_{2i} - z_{2i-1}}$$

Step 4 For $i = 2, 3, \ldots, 2n+1$

for $j = 2, 3, \ldots, i$ set $Q_{i,j} = \dfrac{Q_{i,j-1} - Q_{i-1,j-1}}{z_i - z_{i-j}}$

Step 5 OUTPUT $(Q_{0,0}, Q_{1,1}, \ldots, Q_{2n+1,2n+1})$
STOP

Questions?

Reference Material

Divided Diffences & Derivatives

Theorem

- Suppose that $f \in C^n[a, b]$ and $x_0, x_1, \ldots, x_n$ are distinct numbers in $[a, b]$.
- Then a number ξ exists in (a, b) with

$$f[x_0, x_1, \ldots, x_n] = \frac{f^{(n)}(\xi)}{n!}.$$

See Prrof

Return to Hermite Polynomials & Divided Differences Introduction

Return to Hermite Polynomials & Divided Differences Construction

Divided Diffences & Derivatives: Proof

Let $g(x) = f(x) - P_n(x)$. Since $f(x_i) = P_n(x_i)$ for each $i = 0, 1, \ldots, n$, the function g has $n+1$ distinct zeros in $[a, b]$. The Generalized Rolle's Theorem ▸ Theorem implies that a number ξ in (a, b) exists with $g^{(n)}(\xi) = 0$, so

$$0 = f^{(n)}(\xi) - P_n^{(n)}(\xi).$$

Since $P_n(x)$ is a polynomial of degree n whose leading coefficient is $f[x_0, x_1, \ldots, x_n]$,

$$P_n^{(n)}(x) = n! f[x_0, x_1, \ldots, x_n],$$

for all values of x. As a consequence,

$$f[x_0, x_1, \ldots, x_n] = \frac{f^{(n)}(\xi)}{n!}.$$

▸ Return to Theorem Statement

Generalized Rolle's Theorem

Suppose $f \in C[a, b]$ is n times differentiable on (a, b). If

$$f(x) = 0$$

at the $n + 1$ distinct numbers $a \leq x_0 < x_1 < \ldots < x_n \leq b$, then a number c in (x_0, x_n), and hence in (a, b), exists with

$$f^{(n)}(c) = 0$$

Return to Dividede Difference Proof

Generating the Divided Difference Table

x	$f(x)$	First divided differences	Second divided differences	Third divided differences
x_0	$f[x_0]$			
		$f[x_0, x_1] = \dfrac{f[x_1] - f[x_0]}{x_1 - x_0}$		
x_1	$f[x_1]$		$f[x_0, x_1, x_2] = \dfrac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$	
		$f[x_1, x_2] = \dfrac{f[x_2] - f[x_1]}{x_2 - x_1}$		$f[x_0, x_1, x_2, x_3] = \dfrac{f[x_1, x_2, x_3] - f[x_0, x_1, x_2]}{x_3 - x_0}$
x_2	$f[x_2]$		$f[x_1, x_2, x_3] = \dfrac{f[x_2, x_3] - f[x_1, x_2]}{x_3 - x_1}$	
		$f[x_2, x_3] = \dfrac{f[x_3] - f[x_2]}{x_3 - x_2}$		$f[x_1, x_2, x_3, x_4] = \dfrac{f[x_2, x_3, x_4] - f[x_1, x_2, x_3]}{x_4 - x_1}$
x_3	$f[x_3]$		$f[x_2, x_3, x_4] = \dfrac{f[x_3, x_4] - f[x_2, x_3]}{x_4 - x_2}$	
		$f[x_3, x_4] = \dfrac{f[x_4] - f[x_3]}{x_4 - x_3}$		$f[x_2, x_3, x_4, x_5] = \dfrac{f[x_3, x_4, x_5] - f[x_2, x_3, x_4]}{x_5 - x_2}$
x_4	$f[x_4]$		$f[x_3, x_4, x_5] = \dfrac{f[x_4, x_5] - f[x_3, x_4]}{x_5 - x_3}$	
		$f[x_4, x_5] = \dfrac{f[x_5] - f[x_4]}{x_5 - x_4}$		
x_5	$f[x_5]$			

Return to Hermite Polynomials II