Numerical Differentiation & Integration

Richardson's Extrapolation

Numerical Analysis (9th Edition)

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Outline

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1. Extrapolation & Truncation Errors
2. Example: Improving first order approximations

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2. Example: Improving first order approximations
3. Truncation errors with only even powers of h

Outline

Numerical Differentiation: Richardson Extrapolation

When & how can Extrapolation be Applied

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- Extrapolation can be applied whenever it is known that an approximation technique has an error term with a predictable form, one that depends on a parameter, usually the step size h.

Numerical Differentiation: Richardson Extrapolation

When & how can Extrapolation be Applied

- Richardson's extrapolation is used to generate high-accuracy results while using low-order formulas.
- Extrapolation can be applied whenever it is known that an approximation technique has an error term with a predictable form, one that depends on a parameter, usually the step size h.
- Suppose that, for each number $h \neq 0$, we have a formula $N_1(h)$ that approximates an unknown constant M, and that the truncation error involved with the approximation has the form

$$M - N_1(h) = K_1 h + K_2 h^2 + K_3 h^3 + \cdots$$

for some collection of (unknown) constants $K_1, K_2, K_3, \ldots$.

Numerical Differentiation: Richardson Extrapolation

$$M - N_1(h) = K_1 h + K_2 h^2 + K_3 h^3 + \cdots$$

When & how can Extrapolation be Applied (Cont'd)

Numerical Differentiation: Richardson Extrapolation

$$M - N_1(h) = K_1 h + K_2 h^2 + K_3 h^3 + \cdots$$

When & how can Extrapolation be Applied (Cont'd)

- The truncation error is $O(h)$, so unless there was a large variation in magnitude among the constants $K_1, K_2, K_3, \ldots,$

$$M - N_1(0.1) \approx 0.1K_1, \qquad M - N_1(0.01) \approx 0.01K_1$$

and, in general, $M - N_1(h) \approx K_1 h$.

Numerical Differentiation: Richardson Extrapolation

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When & how can Extrapolation be Applied (Cont'd)

- The truncation error is $O(h)$, so unless there was a large variation in magnitude among the constants $K_1, K_2, K_3, \ldots$,

$$M - N_1(0.1) \approx 0.1K_1, \qquad M - N_1(0.01) \approx 0.01K_1$$

  and, in general, $M - N_1(h) \approx K_1 h$.
- The object of extrapolation is to find an easy way to combine these rather inaccurate $O(h)$ approximations in an appropriate way to produce formulas with a higher-order truncation error.

Numerical Differentiation: Richardson Extrapolation

When & how can Extrapolation be Applied (Cont'd)

Numerical Differentiation: Richardson Extrapolation

When & how can Extrapolation be Applied (Cont'd)

- Suppose, for example, we can combine the $N_1(h)$ formulas to produce an $O(h^2)$ approximation formula, $N_2(h)$, for M with

$$M - N_2(h) = \hat{K}_2 h^2 + \hat{K}_3 h^3 + \cdots$$

for some, again unknown, collection of constants $\hat{K}_2, \hat{K}_3, \ldots$.

Numerical Differentiation: Richardson Extrapolation

When & how can Extrapolation be Applied (Cont'd)

- Suppose, for example, we can combine the $N_1(h)$ formulas to produce an $O(h^2)$ approximation formula, $N_2(h)$, for M with

$$M - N_2(h) = \hat{K}_2 h^2 + \hat{K}_3 h^3 + \cdots$$

for some, again unknown, collection of constants $\hat{K}_2, \hat{K}_3, \ldots$.

- Then we would have

$$M - N_2(0.1) \approx 0.01\hat{K}_2, \qquad M - N_2(0.01) \approx 0.0001\hat{K}_2$$

and so on.

Numerical Differentiation: Richardson Extrapolation

$$M - N_1(h) = K_1 h + K_2 h^2 + K_3 h^3 + \cdots$$

$$M - N_2(h) = \hat{K}_2 h^2 + \hat{K}_3 h^3 + \cdots$$

When & how can Extrapolation be Applied (Cont'd)

Numerical Differentiation: Richardson Extrapolation

$$
\begin{aligned}
M - N_1(h) &= K_1 h + K_2 h^2 + K_3 h^3 + \cdots \\
M - N_2(h) &= \hat{K}_2 h^2 + \hat{K}_3 h^3 + \cdots
\end{aligned}
$$

When & how can Extrapolation be Applied (Cont'd)

- If the constants K_1 and $\hat{K}_2$ are roughly of the same magnitude, then the $N_2(h)$ approximations would be much better than the corresponding $N_1(h)$ approximations.

Numerical Differentiation: Richardson Extrapolation

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When & how can Extrapolation be Applied (Cont'd)

- If the constants K_1 and $\hat{K}_2$ are roughly of the same magnitude, then the $N_2(h)$ approximations would be much better than the corresponding $N_1(h)$ approximations.
- The extrapolation continues by combining the $N_2(h)$ approximations in a manner that produces formulas with $O(h^3)$ truncation error, and so on.

Numerical Differentiation: Richardson Extrapolation

Generating the Extrapolation Formula

Numerical Differentiation: Richardson Extrapolation

Generating the Extrapolation Formula

- To see specifically how we can generate the extrapolation formulas, consider the $O(h)$ formula for approximating M

$$M = N_1(h) + K_1 h + K_2 h^2 + K_3 h^3 + \cdots$$

Numerical Differentiation: Richardson Extrapolation

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- The formula is assumed to hold for all positive h, so we replace the parameter h by half its value.

Numerical Differentiation: Richardson Extrapolation

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- To see specifically how we can generate the extrapolation formulas, consider the $O(h)$ formula for approximating M

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- The formula is assumed to hold for all positive h, so we replace the parameter h by half its value.
- Then we have a second $O(h)$ approximation formula

$$M = N_1\left(\frac{h}{2}\right) + K_1 \frac{h}{2} + K_2 \frac{h^2}{4} + K_3 \frac{h^3}{8} + \cdots$$

Numerical Differentiation: Richardson Extrapolation

$$M = N_1(h) + K_1 h + K_2 h^2 + K_3 h^3 + \cdots$$

$$M = N_1\left(\frac{h}{2}\right) + K_1 \frac{h}{2} + K_2 \frac{h^2}{4}$$

Generating the Extrapolation Formula (Cont'd)

Numerical Differentiation: Richardson Extrapolation

$$M = N_1(h) + K_1 h + K_2 h^2 + K_3 h^3 + \cdots$$

$$M = N_1\left(\frac{h}{2}\right) + K_1 \frac{h}{2} + K_2 \frac{h^2}{4}$$

Generating the Extrapolation Formula (Cont'd)

Subtracting the first from twice the second eliminates the term involving K_1 and gives

Numerical Differentiation: Richardson Extrapolation

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Generating the Extrapolation Formula (Cont'd)

Subtracting the first from twice the second eliminates the term involving K_1 and gives

$$M = N_1\left(\frac{h}{2}\right) + \left[N_1\left(\frac{h}{2}\right) - N_2(h)\right] + K_2\left(\frac{h^2}{2} - h^2\right) + K_3\left(\frac{h^3}{4} - h^3\right) + \cdots$$

Numerical Differentiation: Richardson Extrapolation

$$M = N_1\left(\frac{h}{2}\right) + \left[N_1\left(\frac{h}{2}\right) - N_2(h)\right] + K_2\left(\frac{h^2}{2} - h^2\right) + K_3\left(\frac{h^3}{4} - h^3\right) + \cdots$$

Generating the Extrapolation Formula (Cont’d)

Numerical Differentiation: Richardson Extrapolation

$$M = N_1\left(\frac{h}{2}\right) + \left[N_1\left(\frac{h}{2}\right) - N_2(h)\right] + K_2\left(\frac{h^2}{2} - h^2\right) + K_3\left(\frac{h^3}{4} - h^3\right) + \cdots$$

Generating the Extrapolation Formula (Cont'd)

- Define

$$N_2(h) = N_1\left(\frac{h}{2}\right) + \left[N_1\left(\frac{h}{2}\right) - N_1(h)\right]$$

Numerical Differentiation: Richardson Extrapolation

$$M = N_1\left(\frac{h}{2}\right) + \left[N_1\left(\frac{h}{2}\right) - N_2(h)\right] + K_2\left(\frac{h^2}{2} - h^2\right) + K_3\left(\frac{h^3}{4} - h^3\right) + \cdots$$

Generating the Extrapolation Formula (Cont'd)

- Define
$$N_2(h) = N_1\left(\frac{h}{2}\right) + \left[N_1\left(\frac{h}{2}\right) - N_1(h)\right]$$
- Then the above equation is an $O(h^2)$ approximation formula for M:
$$M = N_2(h) - \frac{K_2}{2}h^2 - \frac{3K_3}{4}h^3 - \cdots$$

Outline

Numerical Differentiation: Richardson Extrapolation

Example: $f(x) = \ln x$

- In an earlier example, we used the forward-difference method

$$f'(x_0) = \frac{f(x_0+h) - f(x_0)}{h} - \frac{h}{2}f''(\xi)$$

with $h = 0.1$ and $h = 0.05$ to find approximations to $f'(1.8)$ for $f(x) = \ln(x)$.

Numerical Differentiation: Richardson Extrapolation

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  with $h = 0.1$ and $h = 0.05$ to find approximations to $f'(1.8)$ for $f(x) = \ln(x)$.

- Assume that this formula has truncation error $O(h)$ and use extrapolation on these values to see if this results in a better approximation.

Numerical Differentiation: Richardson Extrapolation

Solution

Using the forward-difference method

$$f'(x_0) \approx \frac{f(x_0+h) - f(x_0)}{h}$$

Numerical Differentiation: Richardson Extrapolation

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Numerical Differentiation: Richardson Extrapolation

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Using the forward-difference method

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we find that

$$\begin{aligned} \text{with } h = 0.1: &\qquad f'(1.8) \approx 0.5406722 \\ \text{with } h = 0.05: &\qquad f'(1.8) \approx 0.5479795 \end{aligned}$$

Numerical Differentiation: Richardson Extrapolation

Solution

Using the forward-difference method

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we find that

$$\text{with } h = 0.1: \qquad f'(1.8) \approx 0.5406722$$
$$\text{with } h = 0.05: \qquad f'(1.8) \approx 0.5479795$$

This implies that

$$N_1(0.1) = 0.5406722 \qquad \text{and} \qquad N_1(0.05) = 0.5479795$$

Numerical Differentiation: Richardson Extrapolation

$$N_2(h) = N_1\left(\frac{h}{2}\right) + \left[N_1\left(\frac{h}{2}\right) - N_1(h)\right]$$

Solution (Cont'd)

- Extrapolating these results gives the new approximation

$$N_2(0.1) = N_1(0.05) + (N_1(0.05) - N_1(0.1))$$

Numerical Differentiation: Richardson Extrapolation

$$N_2(h) = N_1\left(\frac{h}{2}\right) + \left[N_1\left(\frac{h}{2}\right) - N_1(h)\right]$$

Solution (Cont'd)

- Extrapolating these results gives the new approximation

$$\begin{aligned} N_2(0.1) &= N_1(0.05) + (N_1(0.05) - N_1(0.1)) \\ &= 0.5479795 + (0.5479795 - 0.5406722) = 0.555287 \end{aligned}$$

Numerical Differentiation: Richardson Extrapolation

$$N_2(h) = N_1\left(\frac{h}{2}\right) + \left[N_1\left(\frac{h}{2}\right) - N_1(h)\right]$$

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- The $h = 0.1$ and $h = 0.05$ results were found to be accurate to within 1.5×10^{-2} and 7.7×10^{-3}, respectively.

Numerical Differentiation: Richardson Extrapolation

$$N_2(h) = N_1\left(\frac{h}{2}\right) + \left[N_1\left(\frac{h}{2}\right) - N_1(h)\right]$$

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- Extrapolating these results gives the new approximation

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- The $h = 0.1$ and $h = 0.05$ results were found to be accurate to within 1.5×10^{-2} and 7.7×10^{-3}, respectively.

- Because $f'(1.8) = 1/1.8 = 0.\overline{5}$, the extrapolated value is accurate to within 2.7×10^{-4}.

Outline

Numerical Differentiation: Richardson Extrapolation

When can be extrapolation applied?

Numerical Differentiation: Richardson Extrapolation

When can be extrapolation applied?

Extrapolation can be applied whenever the truncation error for a formula has the form

$$\sum_{j=1}^{m-1} K_j h^{\alpha_j} + O(h^{\alpha_m})$$

for a collection of constants K_j and when $\alpha_1 < \alpha_2 < \alpha_3 < \cdots < \alpha_m$.

Numerical Differentiation: Richardson Extrapolation

Even Powers of h

Numerical Differentiation: Richardson Extrapolation

Even Powers of h

- Many formulas used for extrapolation have truncation errors that contain only even powers of h, that is, have the form

$$M = N_1(h) + K_1 h^2 + K_2 h^4 + K_3 h^6 + \cdots$$

Numerical Differentiation: Richardson Extrapolation

Even Powers of h

- Many formulas used for extrapolation have truncation errors that contain only even powers of h, that is, have the form

$$M = N_1(h) + K_1h^2 + K_2h^4 + K_3h^6 + \cdots$$

- The extrapolation is much more effective than when all powers of h are present because the averaging process produces results with errors

$$O(h^2),\ \ O(h^4),\ \ O(h^6),\ \ \ldots$$

with essentially no increase in computation, over the results with errors, $O(h)$, $O(h^2)$, $O(h^3)$, $\ldots$.

Richardson Extrapolation: Even Powers of h

Building $O\left(h^{2j}\right)$ Approximations

Richardson Extrapolation: Even Powers of h

Building $O\left(h^{2j}\right)$ Approximations

Assume that approximation has the form

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Richardson Extrapolation: Even Powers of h

Building $O\left(h^{2j}\right)$ Approximations

Assume that approximation has the form

$$M = N_1(h) + K_1 h^2 + K_2 h^4 + K_3 h^6 + \cdots$$

Replacing h with $h/2$ gives the $O(h^2)$ approximation formula

$$M = N_1\left(\frac{h}{2}\right) + K_1 \frac{h^2}{4} + K_2 \frac{h^4}{16} + K_3 \frac{h^6}{64} + \cdots .$$

Richardson Extrapolation: Even Powers of h

Building $O\left(h^{2j}\right)$ Approximations

Assume that approximation has the form

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Replacing h with $h/2$ gives the $O(h^2)$ approximation formula

$$M = N_1\left(\frac{h}{2}\right) + K_1\frac{h^2}{4} + K_2\frac{h^4}{16} + K_3\frac{h^6}{64} + \cdots .$$

Subtracting the first equation from 4 times the second eliminates the h^2 term,

$$3M = \left[4N_1\left(\frac{h}{2}\right) - N_1(h)\right] + K_2\left(\frac{h^4}{4} - h^4\right) + K_3\left(\frac{h^6}{16} - h^6\right) + \cdots$$

Richardson Extrapolation: Even Powers of h

$$3M = \left[4N_1\left(\frac{h}{2}\right) - N_1(h)\right] + K_2\left(\frac{h^4}{4} - h^4\right) + K_3\left(\frac{h^6}{16} - h^6\right) + \cdots$$

Building $O\left(h^{2j}\right)$ Approximations (Cont'd)

Richardson Extrapolation: Even Powers of h

$$3M = \left[4N_1\left(\frac{h}{2}\right) - N_1(h)\right] + K_2\left(\frac{h^4}{4} - h^4\right) + K_3\left(\frac{h^6}{16} - h^6\right) + \cdots$$

Building $O\left(h^{2j}\right)$ Approximations (Cont'd)

Dividing this equation by 3 produces an $O(h^4)$ formula

$$M = \frac{1}{3}\left[4N_1\left(\frac{h}{2}\right) - N_1(h)\right] + \frac{K_2}{3}\left(\frac{h^4}{4} - h^4\right) + \frac{K_3}{3}\left(\frac{h^6}{16} - h^6\right) + \cdots$$

Richardson Extrapolation: Even Powers of h

$$3M = \left[4N_1\left(\frac{h}{2}\right) - N_1(h)\right] + K_2\left(\frac{h^4}{4} - h^4\right) + K_3\left(\frac{h^6}{16} - h^6\right) + \cdots$$

Building $O\left(h^{2j}\right)$ Approximations (Cont'd)

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Define

$$N_2(h) = \frac{1}{3}\left[4N_1\left(\frac{h}{2}\right) - N_1(h)\right] = N_1\left(\frac{h}{2}\right) + \frac{1}{3}\left[N_1\left(\frac{h}{2}\right) - N_1(h)\right]$$

Richardson Extrapolation: Even Powers of h

$$M = \frac{1}{3}\left[4N_1\left(\frac{h}{2}\right) - N_1(h)\right] - K_2\frac{h^4}{4} - K_3\frac{5h^6}{16} + \cdots$$

$$N_2(h) = \frac{1}{3}\left[4N_1\left(\frac{h}{2}\right) - N_1(h)\right] = N_1\left(\frac{h}{2}\right) + \frac{1}{3}\left[N_1\left(\frac{h}{2}\right) - N_1(h)\right]$$

Building $O\left(h^{2j}\right)$ Approximations (Cont'd)

Richardson Extrapolation: Even Powers of h

$$M = \frac{1}{3}\left[4N_1\left(\frac{h}{2}\right) - N_1(h)\right] - K_2\frac{h^4}{4} - K_3\frac{5h^6}{16} + \cdots$$

$$N_2(h) = \frac{1}{3}\left[4N_1\left(\frac{h}{2}\right) - N_1(h)\right] = N_1\left(\frac{h}{2}\right) + \frac{1}{3}\left[N_1\left(\frac{h}{2}\right) - N_1(h)\right]$$

Building $O\left(h^{2j}\right)$ Approximations (Cont'd)

With this definition of $N_2(h)$, we obtain the approximation formula with truncation error $O(h^4)$

$$M = N_2(h) - K_2\frac{h^4}{4} - K_3\frac{5h^6}{16} + \cdots$$

Richardson Extrapolation: Even Powers of h

Building $O\left(h^{2j}\right)$ Approximations (Cont'd)

Now replace h in this new $O(h^4)$ formula

$$M = N_2(h) - K_2\frac{h^4}{4} - K_3\frac{5h^6}{16} + \cdots$$

with $h/2$

Richardson Extrapolation: Even Powers of h

Building $O\left(h^{2j}\right)$ Approximations (Cont'd)

Now replace h in this new $O(h^4)$ formula

$$M = N_2(h) - K_2\frac{h^4}{4} - K_3\frac{5h^6}{16} + \cdots$$

with $h/2$ to produce a second $O(h^4)$ formula

$$M = N_2\left(\frac{h}{2}\right) - K_2\frac{5h^4}{64} - K_3\frac{h^6}{1024} - \cdots$$

Richardson Extrapolation: Even Powers of h

Building $O\left(h^{2j}\right)$ Approximations (Cont'd)

Now replace h in this new $O(h^4)$ formula

$$M = N_2(h) - K_2 \frac{h^4}{4} - K_3 \frac{5h^6}{16} + \cdots$$

with $h/2$ to produce a second $O(h^4)$ formula

$$M = N_2\left(\frac{h}{2}\right) - K_2 \frac{5h^4}{64} - K_3 \frac{h^6}{1024} - \cdots$$

Subtracting the first equation from 16 times the second eliminates the h^4 term and gives

$$15M = \left[16N_2\left(\frac{h}{2}\right) - N_2(h)\right] + K_3 \frac{15h^6}{64} + \cdots$$

Richardson Extrapolation: Even Powers of h

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Building $O\left(h^{2j}\right)$ Approximations (Cont'd)

Dividing this equation by 15

Richardson Extrapolation: Even Powers of h

$$15M = \left[16N_2\left(\frac{h}{2}\right) - N_2(h)\right] + K_3\frac{15h^6}{64} + \cdots$$

Building $O\left(h^{2j}\right)$ Approximations (Cont'd)

Dividing this equation by 15 produces the new $O(h^6)$ formula

$$M = \frac{1}{15}\left[16N_2\left(\frac{h}{2}\right) - N_2(h)\right] + K_3\frac{h^6}{64} + \cdots$$

Richardson Extrapolation: Even Powers of h

$$15M = \left[16N_2\left(\frac{h}{2}\right) - N_2(h)\right] + K_3\frac{15h^6}{64} + \cdots$$

Building $O\left(h^{2j}\right)$ Approximations (Cont'd)

Dividing this equation by 15 produces the new $O(h^6)$ formula

$$M = \frac{1}{15}\left[16N_2\left(\frac{h}{2}\right) - N_2(h)\right] + K_3\frac{h^6}{64} + \cdots$$

We now have the $O(h^6)$ approximation formula

$$N_3(h) = \frac{1}{15}\left[16N_2\left(\frac{h}{2}\right) - N_2(h)\right] = N_2\left(\frac{h}{2}\right) + \frac{1}{15}\left[N_2\left(\frac{h}{2}\right) - N_2(h)\right]$$

Richardson Extrapolation: Even Powers of h

Building $O\left(h^{2j}\right)$ Approximations (Cont'd)

Continuing this procedure gives, for each $j = 2, 3, \ldots$, the $O(h^{2j})$ approximation

$$N_j(h) = N_{j-1}\left(\frac{h}{2}\right) + \frac{N_{j-1}(h/2) - N_{j-1}(h)}{4^{j-1} - 1}$$

Richardson Extrapolation: Even Powers of h

Building $O\left(h^{2j}\right)$ Approximations (Cont'd)

Continuing this procedure gives, for each $j = 2, 3, \ldots$, the $O(h^{2j})$ approximation

$$N_j(h) = N_{j-1}\left(\frac{h}{2}\right) + \frac{N_{j-1}(h/2) - N_{j-1}(h)}{4^{j-1} - 1}$$

The table on the next slide shows the order in which the approximations are generated when

$$M = N_1(h) + K_1 h^2 + K_2 h^4 + K_3 h^6 + \cdots$$

Richardson Extrapolation: Even Powers of h

Building $O\left(h^{2j}\right)$ Approximations (Cont'd)

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$$N_j(h) = N_{j-1}\left(\frac{h}{2}\right) + \frac{N_{j-1}(h/2) - N_{j-1}(h)}{4^{j-1} - 1}$$

The table on the next slide shows the order in which the approximations are generated when

$$M = N_1(h) + K_1 h^2 + K_2 h^4 + K_3 h^6 + \cdots$$

It is conservatively assumed that the true result is accurate at least to within the agreement of the bottom two results in the diagonal, in this case, to within $|N_3(h) - N_4(h)|$.

Richardson Extrapolation: Even Powers of h

$$N_j(h) = N_{j-1}\left(\frac{h}{2}\right) + \frac{N_{j-1}(h/2) - N_{j-1}(h)}{4^{j-1} - 1}.$$

Tabular Generation of the $O\left(h^{2j}\right)$ Approximations

$O(h^2)$	$O(h^4)$	$O(h^6)$	$O(h^8)$
1: $N_1(h)$			
2: $N_1(\frac{h}{2})$	**3:** $N_2(h)$		
4: $N_1(\frac{h}{4})$	**5:** $N_2(\frac{h}{2})$	**6:** $N_3(h)$	
7: $N_1(\frac{h}{8})$	**8:** $N_2(\frac{h}{4})$	**9:** $N_3(\frac{h}{2})$	**10:** $N_4(h)$

Richardson Extrapolation: Even Powers of h

Example: Centered-difference Formula

Taylor's theorem can be used to show that the centered-difference formula to approximate $f'(x_0)$

$$f'(x_0) = \frac{1}{2h}[f(x_0+h) - f(x_0-h)] - \frac{h^2}{6}f^{(3)}(\xi_1)$$

(where ξ_1 lies between $x_0 - h$ and $x_0 + h$)

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(where ξ_1 lies between $x_0 - h$ and $x_0 + h$) can be expressed with an error formula:

$$f'(x_0) = \frac{1}{2h}[f(x_0+h) - f(x_0-h)] - \frac{h^2}{6}f'''(x_0) - \frac{h^4}{120}f^{(5)}(x_0) - \cdots$$

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Find approximations of order $O(h^2)$, $O(h^4)$, and $O(h^6)$ for $f'(2.0)$ when $f(x) = xe^x$ and $h = 0.2$.

Richardson Extrapolation: Even Powers of h

Solution (1 of 6)

- The constants $K_1 = -f'''(x_0)/6$, $K_2 = -f^{(5)}(x_0)/120, \cdots$, are not likely to be known, but this is not important.

Richardson Extrapolation: Even Powers of h

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Richardson Extrapolation: Even Powers of h

Solution (1 of 6)

- The constants $K_1 = -f'''(x_0)/6$, $K_2 = -f^{(5)}(x_0)/120, \cdots$, are not likely to be known, but this is not important.
- We only need to know that these constants exist in order to apply extrapolation.
- We have the $O(h^2)$ approximation

$$f'(x_0) = N_1(h) - \frac{h^2}{6}f'''(x_0) - \frac{h^4}{120}f^{(5)}(x_0) - \cdots$$

where

$$N_1(h) = \frac{1}{2h}[f(x_0 + h) - f(x_0 - h)]$$

Richardson Extrapolation: Even Powers of h

$$N_1(h) = \frac{1}{2h}[f(x_0 + h) - f(x_0 - h)]$$

Solution (2 of 6)

Richardson Extrapolation: Even Powers of h

$$N_1(h) = \frac{1}{2h}[f(x_0 + h) - f(x_0 - h)]$$

Solution (2 of 6)

This gives us the first $O(h^2)$ approximations

$$N_1(0.2) = \frac{1}{0.4}[f(2.2) - f(1.8)] = 2.5(19.855030 - 10.889365)$$

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Richardson Extrapolation: Even Powers of h

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&= 22.414160 \\
N_1(0.1) &= \frac{1}{0.2}[f(2.1) - f(1.9)] = 5(17.148957 - 12.703199) \\
&= 22.228786
\end{aligned}$$

Richardson Extrapolation: Even Powers of h

$$N_j(h) = N_{j-1}\left(\frac{h}{2}\right) + \frac{N_{j-1}(h/2) - N_{j-1}(h)}{4^{j-1} - 1}.$$

Solution (3 of 6)

Richardson Extrapolation: Even Powers of h

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Combining these to produce the first $O(h^4)$ approximation gives

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Solution (3 of 6)

Combining these to produce the first $O(h^4)$ approximation gives

$$\begin{aligned} N_2(0.2) &= N_1(0.1) + \frac{1}{3}(N_1(0.1) - N_1(0.2)) \\ &= 22.228786 + \frac{1}{3}(22.228786 - 22.414160) = 22.166995 \end{aligned}$$

Richardson Extrapolation: Even Powers of h

Solution (4 of 6)

To determine an $O(h^6)$ formula, we need another $O(h^4)$ result,

Richardson Extrapolation: Even Powers of h

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To determine an $O(h^6)$ formula, we need another $O(h^4)$ result, which requires us to find the third $O(h^2)$ approximation:

$$N_1(0.05) = \frac{1}{0.1}[f(2.05) - f(1.95)]$$

Richardson Extrapolation: Even Powers of h

Solution (4 of 6)

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$$\begin{aligned} N_1(0.05) &= \frac{1}{0.1}[f(2.05) - f(1.95)] \\ &= 10(15.924197 - 13.705941) = 22.182564 \end{aligned}$$

Richardson Extrapolation: Even Powers of h

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We can now find the $O(h^4)$ approximation

$$N_2(0.1) = N_1(0.05) + \frac{1}{3}(N_1(0.05) - N_1(0.1))$$

Richardson Extrapolation: Even Powers of h

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We can now find the $O(h^4)$ approximation

$$\begin{aligned} N_2(0.1) &= N_1(0.05) + \frac{1}{3}(N_1(0.05) - N_1(0.1)) \\ &= 22.182564 + \frac{1}{3}(22.182564 - 22.228786) = 22.167157 \end{aligned}$$

Richardson Extrapolation: Even Powers of h

$$N_j(h) = N_{j-1}\left(\frac{h}{2}\right) + \frac{N_{j-1}(h/2) - N_{j-1}(h)}{4^{j-1} - 1}.$$

Solution (5 of 6)

Richardson Extrapolation: Even Powers of h

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Finally, we can compute the $O(h^6)$ approximation

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Richardson Extrapolation: Even Powers of h

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Solution (5 of 6)

Finally, we can compute the $O(h^6)$ approximation

$$\begin{aligned} N_3(0.2) &= N_2(0.1) + \frac{1}{15}(N_2(0.1) - N_1(0.2)) \\ &= 22.167157 + \frac{1}{15}(22.167157 - 22.166995) = 22.167168 \end{aligned}$$

Richardson Extrapolation: Even Powers of h

Solution (6 of 6)

- We would expect the final approximation to be accurate to at least the value 22.167 because the $N_2(0.2)$ and $N_3(0.2)$ give this same value.

Richardson Extrapolation: Even Powers of h

Solution (6 of 6)

- We would expect the final approximation to be accurate to at least the value 22.167 because the $N_2(0.2)$ and $N_3(0.2)$ give this same value.
- In fact, $N_3(0.2)$ is accurate to all the listed digits.

Richardson Extrapolation: Danger of Round-Off Errors

Ensuring accuracy

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- However, as k increases, the round-off error in $N_1(h/2^k)$ will generally increase because the instability of numerical differentiation is related to the step size $h/2^k$.
- Also, the higher-order formulas depend increasingly on the entry to their immediate left in the table, which is the reason we recommend comparing the final diagonal entries to ensure accuracy.

Questions?