Initial-Value Problems for ODEs

Elementary Theory of Initial-Value Problems

Numerical Analysis (9th Edition)

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Outline

1 Lipschitz Condition & Convexity

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1. Lipschitz Condition & Convexity
2. The Existence of a Unique Solution

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3. Well-Posed Problems

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Elementary Theory of IVPs: Lipschitz Condition

We begin by presenting some definitions and results from the theory of ordinary differential equations before considering methods for approximating the solutions to initial-value problems.

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Definition: Lipschitz Condition

A function $f(t, y)$ is said to satisfy a Lipschitz condition in the variable y on a set $D \subset \mathbb{R}^2$ if a constant $L > 0$ exists with

$$|f(t, y_1) - f(t, y_2,)| \leq L\,|y_1 - y_2|$$

whenever (t, y_1) and (t, y_2) are in D. The constant L is called a Lipschitz constant for f.

Elementary Theory of IVPs: Lipschitz Condition

Example

Show that $f(t, y) = t|y|$ satisfies a Lipschitz condition on the interval $D = \{\,(t, y) \mid 1 \leq t \leq 2 \text{ and } -3 \leq y \leq 4\,\}$.

Elementary Theory of IVPs: Lipschitz Condition

Example

Show that $f(t, y) = t|y|$ satisfies a Lipschitz condition on the interval $D = \{(t, y) \mid 1 \le t \le 2 \text{ and } -3 \le y \le 4\}$.

Solution

For each pair of points (t, y_1) and (t, y_2) in D we have

$$|f(t, y_1) - f(t, y_2)| = |t|y_1| - t|y_2|| = |t|\,||y_1| - |y_2|| \le 2|y_1 - y_2|$$

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Thus f satisfies a Lipschitz condition on D in the variable y with Lipschitz constant 2.

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Thus f satisfies a Lipschitz condition on D in the variable y with Lipschitz constant 2. The smallest value possible for the Lipschitz constant for this problem is $L = 2$, because, for example,

$$|f(2, 1) - f(2, 0)| = |2 - 0| = 2|1 - 0|$$

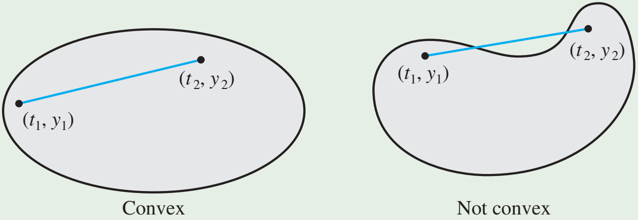
Elementary Theory of IVPs: Convex Set

Definition: Convex Set

A set $D \subset \mathbb{R}^2$ is said to be convex if whenever (t_1, y_1) and (t_2, y_2) belong to D, then

$$((1-\lambda)t_1 + \lambda t_2,\ \ (1-\lambda)y_1 + \lambda y_2)$$

also belongs to D for every λ in $[0, 1]$.



Elementary Theory of IVPs: Convex Set

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- The sets we consider in this section are generally of the form

$$D = \{ (t, y) \mid a \leq t \leq b \quad \text{and} \quad -\infty < y < \infty \}$$

for some constants a and b.

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- In geometric terms, the definition states that a set is convex provided that whenever two points belong to the set, the entire straight-line segment between the points also belongs to the set.
- The sets we consider in this section are generally of the form

$$D = \{\, (t, y) \mid a \leq t \leq b \quad \text{and} \quad -\infty < y < \infty \,\}$$

for some constants a and b.
- It is easy to verify that these sets are convex.

Theory of IVPs: Lipschitz Condition & Convexity

Theorem: Sufficient Conditions

Suppose $f(t, y)$ is defined on a convex set $D \subset \mathbb{R}^2$. If a constant $L > 0$ exists with

$$\left| \frac{\partial f}{\partial y}(t, y) \right| \leq L, \quad \text{for all } (t, y) \in D$$

then f satisfies a Lipschitz condition on D in the variable y with Lipschitz constant L.

Theory of IVPs: Lipschitz Condition & Convexity

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then f satisfies a Lipschitz condition on D in the variable y with Lipschitz constant L.

As the next result will show, this theorem is often of significant interest to determine whether the function involved in an initial-value problem satisfies a Lipschitz condition in its second variable, and the above condition is generally easier to apply than the definition.

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Elementary Theory of IVPs

Theorem: Existence & Uniqueness

Suppose that $D = \{\,(t, y) \mid a \le t \le b \text{ and } -\infty < y < \infty\,\}$ and that $f(t, y)$ is continuous on D. If f satisfies a Lipschitz condition on D in the variable y, then the initial-value problem

$$y'(t) = f(t, y), \quad a \le t \le b, \quad y(a) = \alpha,$$

has a unique solution $y(t)$ for $a \le t \le b$.

Elementary Theory of IVPs

Theorem: Existence & Uniqueness

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has a unique solution $y(t)$ for $a \leq t \leq b$.

Note: This is a version of the fundamental existence and uniqueness theorem for first-order ordinary differential equations. The proof of the theorem, in approximately this form, can be found in Birkhoff, G. and G. Rota, *Ordinary differential equations*, (4th edition), John Wiley & Sons, New York, 1989.

Elementary Theory of IVPs

Example: Applying the Existence & Uniqueness Theorem

Use the Existence & Uniqueness Theorem to show that there is a unique solution to the initial-value problem

$$y' = 1 + t\sin(ty), \qquad 0 \le t \le 2, \qquad y(0) = 0$$

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Solution (1/2)

Holding t constant and applying the Mean Value Theorem ▸ See Theorem to the function

$$f(t, y) = 1 + t\sin(ty)$$

we find that when $y_1 < y_2$, a number ξ in (y_1, y_2) exists with

$$\frac{f(t, y_2) - f(t, y_1)}{y_2 - y_1} = \frac{\partial}{\partial y} f(t, \xi) = t^2\cos(\xi t)$$

Elementary Theory of IVPs

$$\frac{f(t,y_2) - f(t,y_1)}{y_2 - y_1} = \frac{\partial}{\partial y} f(t,\xi) = t^2 \cos(\xi t)$$

Solution (2/2)

Elementary Theory of IVPs

$$\frac{f(t, y_2) - f(t, y_1)}{y_2 - y_1} = \frac{\partial}{\partial y} f(t, \xi) = t^2 \cos(\xi t)$$

Solution (2/2)

- Thus

$$|f(t, y_2) - f(t, y_1)| = |y_2 - y_1||t^2 \cos(\xi t)| \leq 4|y_2 - y_1|$$

and f satisfies a Lipschitz condition in the variable y with Lipschitz constant $L = 4$.

Elementary Theory of IVPs

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  and f satisfies a Lipschitz condition in the variable y with Lipschitz constant $L = 4$.

- Additionally, $f(t,y)$ is continuous when $0 \leq t \leq 2$ and $-\infty < y < \infty$, so the Existence & Uniqueness Theorem implies that a unique solution exists to this initial-value problem.

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Elementary Theory of IVPs: Well-Posed problems

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How do we determine whether a particular problem has the property that small changes, or perturbations, in the statement of the problem introduce correspondingly small changes in the solution?

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We first need to give a workable definition to express this concept.

Elementary Theory of IVPs: Well-Posed Problems

Definition: Well-Posed Problem

The initial-value problem

$$\frac{dy}{dt} = f(t, y), \quad a \leq t \leq b, \quad y(a) = \alpha$$

is said to be a well-posed problem if the following 2 conditions are satisfied:

Elementary Theory of IVPs: Well-Posed Problems

Definition: Well-Posed Problem (Continued)

- A unique solution, $y(t)$, to the problem exists, and

Elementary Theory of IVPs: Well-Posed Problems

Definition: Well-Posed Problem (Continued)

- A unique solution, $y(t)$, to the problem exists, and
- There exist constants $\varepsilon_0 > 0$ and $k > 0$ such that for any ε, with $\varepsilon_0 > \varepsilon > 0$, whenever $\delta(t)$ is continuous with $|\delta(t)| < \varepsilon$ for all t in $[a, b]$, and when $|\delta_0| < \varepsilon$, the initial-value problem

$$\frac{dz}{dt} = f(t, z) + \delta(t), \quad a \leq t \leq b, \quad z(a) = \alpha + \delta_0$$

has a unique solution $z(t)$ that satisfies

$$|z(t) - y(t)| < k\varepsilon \quad \text{for all } t \text{ in } [a, b].$$

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Note: The problem in z, as specified above, is called a perturbed problem associated with the original problem for y.

Elementary Theory of IVPs: Well-Posed Problems

Conditions to ensure that an initial-value problem is well-posed.

Elementary Theory of IVPs: Well-Posed Problems

Conditions to ensure that an initial-value problem is well-posed.

Theorem: Well-Posed Problem

Suppose $D = \{ (t, y) \mid a \leq t \leq b \text{ and } -\infty < y < \infty \}$. If f is continuous and satisfies a Lipschitz condition in the variable y on the set D, then the initial-value problem

$$\frac{dy}{dt} = f(t, y), \quad a \leq t \leq b, \quad y(a) = \alpha$$

is well-posed.

The proof of this theorem can be found in Birkhoff, G. and G. Rota, *Ordinary differential equations*, (4th edition), John Wiley & Sons, New York, 1989.

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Elementary Theory of IVPs: Well-Posed Problems

Example: Applying the Theorem on Well-Posed Problems

Show that the initial-value problem

$$\frac{dy}{dt} = y - t^2 + 1, \quad 0 \leq t \leq 2, \quad y(0) = 0.5$$

is well posed on $D = \{\, (t, y) \mid 0 \leq t \leq 2 \text{ and } -\infty < y < \infty \,\}$.

Elementary Theory of IVPs: Well-Posed Problems

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Solution (1/3)

- Because

$$\left|\frac{\partial(y - t^2 + 1)}{\partial y}\right| = |1| = 1$$

the Lipschitz Condition theorem implies that $f(t, y) = y - t^2 + 1$ satisfies a Lipschitz condition in y on D with Lipschitz constant 1.

Elementary Theory of IVPs: Well-Posed Problems

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the Lipschitz Condition theorem implies that $f(t, y) = y - t^2 + 1$ satisfies a Lipschitz condition in y on D with Lipschitz constant 1.

- Since f is continuous on D, the Theorem on Well-Posed Problems implies that the problem is well-posed.

Elementary Theory of IVPs: Well-Posed Problems

Solution (2/3)

Elementary Theory of IVPs: Well-Posed Problems

Solution (2/3)

- As an illustration, consider the solution to the perturbed problem

$$\frac{dz}{dt} = z - t^2 + 1 + \delta, \quad 0 \le t \le 2, \quad z(0) = 0.5 + \delta_0$$

where δ and δ_0 are constants.

Elementary Theory of IVPs: Well-Posed Problems

Solution (2/3)

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  where δ and δ_0 are constants.

- The solutions to the original problem and this perturbed problem are

$$\begin{aligned} y(t) &= (t+1)^2 - 0.5e^t \\ \text{and} \quad z(t) &= (t+1)^2 + (\delta + \delta_0 - 0.5)e^t - \delta \end{aligned}$$

  respectively.

Elementary Theory of IVPs: Well-Posed Problems

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Solution (3/3)

Elementary Theory of IVPs: Well-Posed Problems

$$\begin{aligned} y(t) &= (t+1)^2 - 0.5e^t \\ z(t) &= (t+1)^2 + (\delta + \delta_0 - 0.5)e^t - \delta \end{aligned}$$

Solution (3/3)

- Suppose that ε is a positive number. If $|\delta| < \varepsilon$ and $|\delta_0| < \varepsilon$, then

$$|y(t) - z(t)| = |(\delta + \delta_0)e^t - \delta| \leq |\delta + \delta_0|e^2 + |\delta| \leq (2e^2 + 1)\varepsilon$$

for all t.

Elementary Theory of IVPs: Well-Posed Problems

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y(t) &= (t+1)^2 - 0.5e^t \\
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Solution (3/3)

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  for all t.
- This implies that the original problem is well-posed with $k(\varepsilon) = 2e^2 + 1$ for all $\varepsilon > 0$.

Questions?

Reference Material

Mean Value Theorem

If $f \in C[a,b]$ and f is differentiable on (a,b), then a number c exists such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$