Initial-Value Problems for ODEs

Euler's Method I: Introduction

Numerical Analysis (9th Edition)

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Outline

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Euler's Method: Derivation

Obtaining Approximations

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Obtaining Approximations

- The object of Euler's method is to obtain approximations to the well-posed initial-value problem

$$\frac{dy}{dt} = f(t, y), \quad a \leq t \leq b, \quad y(a) = \alpha$$

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- A continuous approximation to the solution $y(t)$ will not be obtained;
- Instead, approximations to y will be generated at various values, called mesh points, in the interval $[a, b]$.
- Once the approximate solution is obtained at the points, the approximate solution at other points in the interval can be found by interpolation.

Euler's Method: Derivation (Cont'd

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$$t_i = a + ih, \quad \text{for each } i = 0, 1, 2, \ldots, N.$$

- The common distance between the points $h = (b - a)/N = t_{i+1} - t_i$ is called the step size.

Euler's Method: Derivation (Cont'd

Use Taylor's Theorem to derive Euler's Method

Euler's Method: Derivation (Cont'd

Use Taylor's Theorem to derive Euler's Method

- Suppose that $y(t)$, the unique solution to

$$\frac{dy}{dt} = f(t, y), \quad a \le t \le b, \quad y(a) = \alpha$$

has two continuous derivatives on $[a, b]$, so that for each $i = 0, 1, 2, \ldots, N-1$,

$$y(t_{i+1}) = y(t_i) + (t_{i+1} - t_i)y'(t_i) + \frac{(t_{i+1} - t_i)^2}{2} y''(\xi_i)$$

for some number ξ_i in (t_i, t_{i+1}).

Euler's Method: Derivation (Cont'd

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$$y(t_{i+1}) = y(t_i) + (t_{i+1} - t_i)y'(t_i) + \frac{(t_{i+1} - t_i)^2}{2} y''(\xi_i)$$

- Because $h = t_{i+1} - t_i$, we have

$$y(t_{i+1}) = y(t_i) + hy'(t_i) + \frac{h^2}{2} y''(\xi_i)$$

and, because $y(t)$ satisfies the differential equation $y' = f(t, y)$, we write

$$y(t_{i+1}) = y(t_i) + hf(t_i, y(t_i)) + \frac{h^2}{2} y''(\xi_i)$$

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Euler's Method

Euler's method constructs $w_i \approx y(t_i)$, for each $i = 1, 2, \ldots, N$, by deleting the remainder term. Thus Euler's method is

$$\begin{aligned} w_0 &= \alpha \\ w_{i+1} &= w_i + hf(t_i, w_i), \quad \text{for each } i = 0, 1, \ldots, N-1 \end{aligned}$$

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This equation is called the difference equation associated with Euler's method.

Euler's Method: Illustration

Applying Euler's Method

Prior to introducing an algorithm for Euler's Method, we will illustrate the steps in the technique to approximate the solution to

$$y' = y - t^2 + 1, \quad 0 \leq t \leq 2, \quad y(0) = 0.5$$

at $t = 2$. using a step size of $h = 0.5$.

Euler's Method: Illustration

Solution

For this problem $f(t, y) = y - t^2 + 1$, so

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For this problem $f(t, y) = y - t^2 + 1$, so

$$\begin{aligned} w_0 &= y(0) = 0.5 \\ w_1 &= w_0 + 0.5\left(w_0 - (0.0)^2 + 1\right) = 0.5 + 0.5(1.5) = 1.25 \end{aligned}$$

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\end{aligned}
$$

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\end{aligned}$$

and

$$y(2) \approx w_4 = w_3 + 0.5\left(w_3 - (1.5)^2 + 1\right) = 3.375 + 0.5(2.125) = 4.4375$$

Outline

Euler's Method: Algorithm (1/2)

To approximate the solution of the initial-value problem

$$y' = f(t, y), \quad a \leq t \leq b, \quad y(a) = \alpha$$

at $(N+1)$ equally spaced numbers in the interval $[a, b]$:

Euler's Method: Algorithm (2/2)

INPUT	endpoints a, b; integer N; initial condition α.
OUTPUT	approximation w to y at the $(N+1)$ values of t.
Step 1	Set $h = (b-a)/N$
	$t = a$
	$w = \alpha$
	OUTPUT (t, w)
Step 2	For $i = 1, 2, \ldots, N$ do Steps 3 & 4
	Step 3 Set $w = w + hf(t, w)$; (*Compute* w_i)
	$t = a + ih$. (*Compute* t_i)
	Step 4 OUTPUT (t, w)
Step 5	STOP

Outline

1. Derivation of Euler's Method
2. Numerical Algorithm
3. Geometric Interpretation
4. Numerical Example

Euler's Method: Geometric Interpretation

To interpret Euler's method geometrically, note that when w_i is a close approximation to $y(t_i)$, the assumption that the problem is well-posed implies that

$$f(t_i, w_i) \approx y'(t_i) = f(t_i, y(t_i))$$

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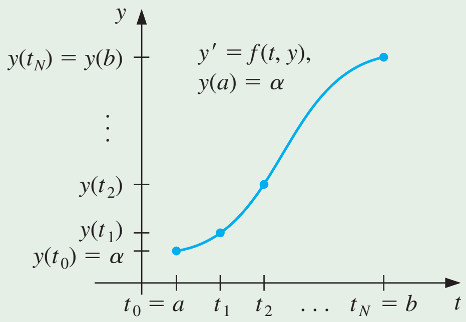
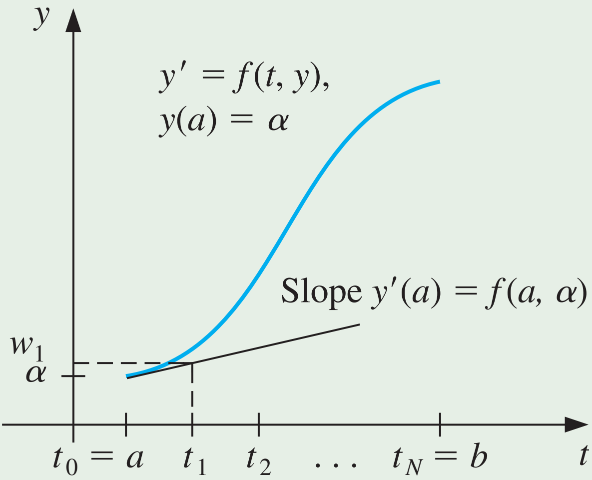
The graph of the function highlighting $y(t_i)$ is shown below.



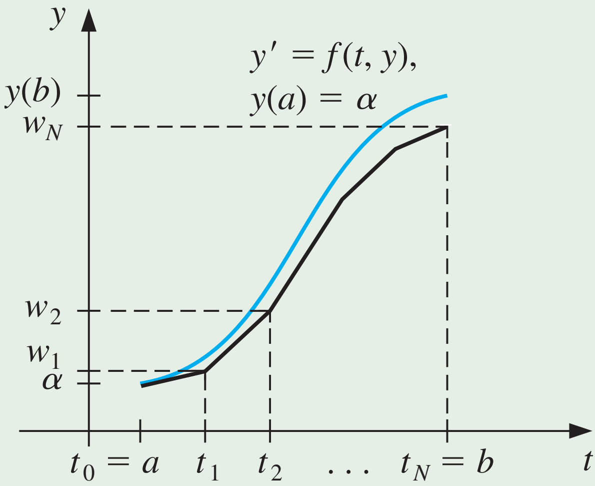
Euler's Method: Geometric Interpretation

One step in Euler's method:



Euler's Method: Geometric Interpretation

A series of steps in Euler's method:



Outline

Euler's Method: Numerical Example (1/4)

Application of Euler's Method

Use the algorithm for Euler's method with $N = 10$ to determine approximations to the solution to the initial-value problem

$$y' = y - t^2 + 1, \quad 0 \leq t \leq 2, \quad y(0) = 0.5$$

and compare these with the exact values given by

$$y(t) = (t + 1)^2 - 0.5e^t$$

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Euler's method constructs $w_i \approx y(t_i)$, for each $i = 1, 2, \ldots, N$:

$$\begin{aligned} w_0 &= \alpha \\ w_{i+1} &= w_i + hf(t_i, w_i), \quad \text{for each } i = 0, 1, \ldots, N-1 \end{aligned}$$

Euler's Method: Numerical Example (2/4)

Solution

Euler's Method: Numerical Example (2/4)

Solution

With $N = 10$, we have $h = 0.2$, $t_i = 0.2i$, $w_0 = 0.5$, so that:

$$w_{i+1} = w_i + h(w_i - t_i^2 + 1)$$

Euler's Method: Numerical Example (2/4)

Solution

With $N = 10$, we have $h = 0.2$, $t_i = 0.2i$, $w_0 = 0.5$, so that:

$$\begin{aligned} w_{i+1} &= w_i + h(w_i - t_i^2 + 1) \\ &= w_i + 0.2[w_i - 0.04i^2 + 1] \end{aligned}$$

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$$\begin{aligned} w_{i+1} &= w_i + h(w_i - t_i^2 + 1) \\ &= w_i + 0.2[w_i - 0.04i^2 + 1] \\ &= 1.2w_i - 0.008i^2 + 0.2 \end{aligned}$$

for $i = 0, 1, \ldots, 9$.

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for $i = 0, 1, \ldots, 9$. So

$$w_1 = 1.2(0.5) - 0.008(0)^2 + 0.2 = 0.8$$

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$$\begin{aligned} w_1 &= 1.2(0.5) - 0.008(0)^2 + 0.2 = 0.8 \\ w_2 &= 1.2(0.8) - 0.008(1)^2 + 0.2 = 1.152 \end{aligned}$$

and so on.

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and so on.

The following table shows the comparison between the approximate values at t_i and the actual values.

Euler's Method: Numerical Example (3/4)

Results for $y' = y - t^2 + 1, \quad 0 \leq t \leq 2, \quad y(0) = 0.5$

t_i	w_i	$y_i = y(t_i)$	$\lvert y_i - w_i \rvert$
0.0	0.5000000	0.5000000	0.0000000
0.2	0.8000000	0.8292986	0.0292986
0.4	1.1520000	1.2140877	0.0620877
0.6	1.5504000	1.6489406	0.0985406
0.8	1.9884800	2.1272295	0.1387495
1.0	2.4581760	2.6408591	0.1826831
1.2	2.9498112	3.1799415	0.2301303
1.4	3.4517734	3.7324000	0.2806266
1.6	3.9501281	4.2834838	0.3333557
1.8	4.4281538	4.8151763	0.3870225
2.0	4.8657845	5.3054720	0.4396874

Euler's Method: Numerical Example (4/4)

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Euler's Method: Numerical Example (4/4)

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- Note that the error grows slightly as the value of t increases.
- This controlled error growth is a consequence of the stability of Euler's method, which implies that the error is expected to grow in no worse than a linear manner.

Questions?