Initial-Value Problems for ODEs

Euler's Method II: Error Bounds

Numerical Analysis (9th Edition)
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Beamer Presentation Slides
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Outline

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1. Computational Lemmas
2. Error Bound for Euler's Method
3. Error Bound Example

Euler's Method: Computational Lemmas

Lemma 1

For all $x \geq -1$ and any positive m, we have

$$0 \leq (1+x)^m \leq e^{mx}$$

Euler's Method: Computational Lemmas

Proof of Lemma 1

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Applying Taylor's Theorem with $f(x) = e^x$, $x_0 = 0$, and $n = 1$ gives

$$e^x = 1 + x + \frac{1}{2}x^2 e^{\xi}$$

where ξ is between x and zero.

Euler's Method: Computational Lemmas

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Euler's Method: Computational Lemmas

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where ξ is between x and zero. Thus

$$0 \leq 1 + x \leq 1 + x + \frac{1}{2}x^2 e^\xi = e^x$$

and, because $1 + x \geq 0$, we have

$$0 \leq (1 + x)^m \leq (e^x)^m = e^{mx}$$

Euler's Method: Computational Lemmas

Lemma 2

If s and t are positive real numbers, $\{a_i\}_{i=0}^{k}$ is a sequence satisfying

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$$a_{i+1} \leq (1+s)a_i + t$$

for each $i = 0, 1, 2, \ldots, k-1$,

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for each $i = 0, 1, 2, \ldots, k-1$, then

$$a_{i+1} \leq e^{(i+1)s}\left(a_0 + \frac{t}{s}\right) - \frac{t}{s}$$

Euler's Method: Computational Lemmas

Proof of Lemma 2 (1/3)

For a fixed integer i, the inequality

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Euler's Method: Computational Lemmas

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Euler's Method: Computational Lemmas

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implies that

$$\begin{aligned} a_{i+1} &\leq (1+s)a_i + t \\ &\leq (1+s)[(1+s)a_{i-1} + t] + t \\ &= (1+s)^2 a_{i-1} + [1 + (1+s)]t \end{aligned}$$

Euler's Method: Computational Lemmas

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implies that

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&\leq (1+s)[(1+s)a_{i-1} + t] + t \\
&= (1+s)^2 a_{i-1} + [1 + (1+s)]t \\
&\leq (1+s)^3 a_{i-2} + \left[1 + (1+s) + (1+s)^2\right] t
\end{aligned}$$

Euler's Method: Computational Lemmas

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For a fixed integer i, the inequality

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implies that

$$\begin{aligned} a_{i+1} &\le (1+s)a_i + t \\ &\le (1+s)[(1+s)a_{i-1} + t] + t \\ &= (1+s)^2 a_{i-1} + [1 + (1+s)]t \\ &\le (1+s)^3 a_{i-2} + \left[1 + (1+s) + (1+s)^2\right] t \\ &\vdots \\ &\le (1+s)^{i+1} a_0 + \left[1 + (1+s) + (1+s)^2 + \cdots + (1+s)^i\right] t \end{aligned}$$

Euler's Method: Computational Lemmas

$$a_{i+1} \leq (1+s)^{i+1} a_0 + \left[1 + (1+s) + (1+s)^2 + \cdots + (1+s)^i\right] t$$

Proof of Lemma 2 (2/3)

Euler's Method: Computational Lemmas

$$a_{i+1} \leq (1+s)^{i+1} a_0 + \left[1 + (1+s) + (1+s)^2 + \cdots + (1+s)^i\right] t$$

Proof of Lemma 2 (2/3)

But

$$1 + (1+s) + (1+s)^2 + \cdots + (1+s)^i = \sum_{j=0}^{i} (1+s)^j$$

is a geometric series with ratio $(1+s)$

Euler's Method: Computational Lemmas

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But

$$1 + (1+s) + (1+s)^2 + \cdots + (1+s)^i = \sum_{j=0}^{i} (1+s)^j$$

is a geometric series with ratio $(1+s)$ that sums to

$$\frac{1-(1+s)^{i+1}}{1-(1+s)} = \frac{1}{s}[(1+s)^{i+1} - 1]$$

Euler's Method: Computational Lemmas

$$a_{i+1} \leq (1+s)^{i+1} a_0 + \left[1 + (1+s) + (1+s)^2 + \cdots + (1+s)^i\right] t$$

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Euler's Method: Computational Lemmas

$$a_{i+1} \le (1+s)^{i+1}a_0 + \left[1 + (1+s) + (1+s)^2 + \cdots + (1+s)^i\right] t$$

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Thus

$$a_{i+1} \le (1+s)^{i+1}a_0 + \frac{(1+s)^{i+1} - 1}{s} t = (1+s)^{i+1}\left(a_0 + \frac{t}{s}\right) - \frac{t}{s}$$

Euler's Method: Computational Lemmas

$$a_{i+1} \le (1+s)^{i+1}a_0 + \left[1 + (1+s) + (1+s)^2 + \cdots + (1+s)^i\right] t$$

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Thus

$$a_{i+1} \le (1+s)^{i+1}a_0 + \frac{(1+s)^{i+1}-1}{s}t = (1+s)^{i+1}\left(a_0 + \frac{t}{s}\right) - \frac{t}{s}$$

and using Lemma 1 with $x = 1+s$ gives

$$a_{i+1} \le e^{(i+1)s}\left(a_0 + \frac{t}{s}\right) - \frac{t}{s}.$$

Outline

Euler's Method: Error Bound Theorem

Theorem

Suppose f is continuous and satisfies a Lipschitz condition with constant L on

$$D = \{ (t, y) \mid a \leq t \leq b \text{ and } -\infty < y < \infty \}$$

and that a constant M exists with

$$|y''(t)| \leq M, \quad \text{for all } t \in [a, b]$$

where $y(t)$ denotes the unique solution to the initial-value problem

$$y' = f(t, y), \quad a \leq t \leq b, \quad y(a) = \alpha$$

Continued on the next slide:

Euler's Method: Error Bound Theorem

Theorem (Cont'd)

Let $w_0, w_1, \ldots, w_N$ be the approximations generated by Euler's method for some positive integer N. Then, for each $i = 0, 1, 2, \ldots, N$,

$$|y(t_i) - w_i| \leq \frac{hM}{2L}\left[e^{L(t_i-a)} - 1\right]$$

Euler’s Method: Error Bound Theorem

Prrof (1/3)

Euler's Method: Error Bound Theorem

Prrof (1/3)

When $i = 0$ the result is clearly true, since $y(t_0) = w_0 = \alpha$. Since $y'(t) = f(t, y)$, we have:

$$y(t_{i+1}) = y(t_i) + hf(t_i, y(t_i)) + \frac{h^2}{2} y''(\xi_i)$$

for $i = 0, 1, \ldots, N-1$. Also, Euler's method is:

$$w_{i+1} = w_i + hf(t_i, w_i)$$

Using the notation $y_i = y(t_i)$ and $y_{i+1} = y(t_{i+1})$, we subtract these two equations to obtain

$$y_{i+1} - w_{i+1} = y_i - w_i + h[f(t_i, y_i) - f(t_i, w_i)] + \frac{h^2}{2} y''(\xi_i)$$

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Hence

$$|y_{i+1} - w_{i+1}| \leq |y_i - w_i| + h|f(t_i, y_i) - f(t_i, w_i)| + \frac{h^2}{2} |y''(\xi_i)|$$

Now f satisfies a Lipschitz condition in the second variable with constant L, and $|y''(t)| \leq M$, so

$$|y_{i+1} - w_{i+1}| \leq (1 + hL)|y_i - w_i| + \frac{h^2 M}{2}$$

Euler's Method: Error Bound Theorem

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Referring to Lemma 2 and letting $s = hL$, $t = h^2M/2$, and $a_j = |y_j - w_j|$, for each $j = 0, 1, \ldots, N$,

Euler's Method: Error Bound Theorem

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Referring to ▸ Lemma 2 and letting $s = hL$, $t = h^2M/2$, and $a_j = |y_j - w_j|$, for each $j = 0, 1, \ldots, N$, we see that

$$|y_{i+1} - w_{i+1}| \leq e^{(i+1)hL} \left(|y_0 - w_0| + \frac{h^2 M}{2hL} \right) - \frac{h^2 M}{2hL}$$

Euler's Method: Error Bound Theorem

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$$|y_{i+1} - w_{i+1}| \leq e^{(i+1)hL}\left(|y_0 - w_0| + \frac{h^2M}{2hL}\right) - \frac{h^2M}{2hL}$$

Because $|y_0 - w_0| = 0$ and $(i+1)h = t_{i+1} - t_0 = t_{i+1} - a$, this implies that

$$|y_{i+1} - w_{i+1}| \leq \frac{hM}{2L}(e^{(t_{i+1}-a)L} - 1)$$

for each $i = 0, 1, \ldots, N - 1$.

Euler's Method: Error Bound Theorem

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- Although this condition often prohibits us from obtaining a realistic error bound, it should be noted that if $\partial f/\partial t$ and $\partial f/\partial y$ both exist, the chain rule for partial differentiation implies that

$$y''(t) = \frac{dy'}{dt}(t) = \frac{df}{dt}(t, y(t)) = \frac{\partial f}{\partial t}(t, y(t)) + \frac{\partial f}{\partial y}(t, y(t)) \cdot f(t, y(t))$$

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- The weakness of the error-bound theorem lies in the requirement that a bound be known for the second derivative of the solution.
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- So it is at times possible to obtain an error bound for $y''(t)$ without explicitly knowing $y(t)$.

Outline

Euler's Method: Error Bound Example

Applying the Theorem

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- The solution to the initial-value problem

$$y' = y - t^2 + 1, \quad 0 \le t \le 2, \quad y(0) = 0.5$$

was approximated in an earlier example using Euler's method with $h = 0.2$.

Euler's Method: Error Bound Example

Applying the Theorem

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  was approximated in an earlier example using Euler's method with $h = 0.2$.
- Use the inequality in the error bound theorem to find bounds for the approximation errors and compare these to the actual errors.

Euler's Method: Error Bound Example

Solution (1/4)

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- Because $f(t, y) = y - t^2 + 1$, we have $\partial f(t, y)/\partial y = 1$ for all y, so $L = 1$.

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- Because $f(t, y) = y - t^2 + 1$, we have $\partial f(t, y)/\partial y = 1$ for all y, so $L = 1$.
- For this problem, the exact solution is $y(t) = (t + 1)^2 - 0.5e^t$, so $y''(t) = 2 - 0.5e^t$ and

$$|y''(t)| \leq 0.5e^2 - 2, \quad \text{for all } t \in [0, 2].$$

Euler's Method: Error Bound Example

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- For this problem, the exact solution is $y(t) = (t+1)^2 - 0.5e^t$, so $y''(t) = 2 - 0.5e^t$ and

$$|y''(t)| \leq 0.5e^2 - 2, \quad \text{for all } t \in [0,2].$$

- Using the inequality in the error bound for Euler's method with $h = 0.2$, $L = 1$, and $M = 0.5e^2 - 2$ gives

$$|y_i - w_i| \leq 0.1(0.5e^2 - 2)(e^{t_i} - 1).$$

Euler's Method: Error Bound Example

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Euler's Method: Error Bound Example

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Solution (2/4)

- Hence

$$|y(0.2) - w_1| \leq 0.1(0.5e^2 - 2)(e^{0.2} - 1) = 0.03752$$

Euler's Method: Error Bound Example

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and so on.

Euler's Method: Error Bound Example

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  and so on.
- The folloiwng table lists the actual error computed in the original example, together with this error bound.

Euler's Method: Error Bound Example

Solution (3/4)

t_i	0.2	0.4	0.6	0.8	1.0
Actual Error	0.02930	0.06209	0.09854	0.13875	0.18268
Error Bound	0.03752	0.08334	0.13931	0.20767	0.29117

t_i	1.2	1.4	1.6	1.8	2.0
Actual Error	0.23013	0.28063	0.33336	0.38702	0.43969
Error Bound	0.39315	0.51771	0.66985	0.85568	1.08264

Euler's Method: Error Bound Example

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- Note that even though the true bound for the second derivative of the solution was used, the error bound is considerably larger than the actual error, especially for increasing values of t.

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- The principal importance of the error-bound formula given in this theorem is that the bound depends linearly on the step size h.

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Solution (4/4)

- Note that even though the true bound for the second derivative of the solution was used, the error bound is considerably larger than the actual error, especially for increasing values of t.
- The principal importance of the error-bound formula given in this theorem is that the bound depends linearly on the step size h.
- Consequently, diminishing the step size should give correspondingly greater accuracy to the approximations.

Questions?

Reference Material

Euler's Method: Computational Lemmas

Lemma 2

If s and t are positive real numbers, $\{a_i\}_{i=0}^{k}$ is a sequence satisfying

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and

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for each $i = 0, 1, 2, \ldots, k-1$, then

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Return to Euler's Error Bound Theorem