# Initial-Value Problems for ODEs

## Higher-Order Taylor Methods

Numerical Analysis (9th Edition)

R L Burden & J D Faires

Beamer Presentation Slides
prepared by
John Carroll
Dublin City University

# Outline

# Outline

# Outline

# Outline

# Outline

# Local Truncation Error

## Informal Definition of LTE

The local truncation error at a specified step measures the amount by which the exact solution to the differential equation fails to satisfy the difference equation being used for the approximation at that step.

# Local Truncation Error

## Informal Definition of LTE

The local truncation error at a specified step measures the amount by which the exact solution to the differential equation fails to satisfy the difference equation being used for the approximation at that step.

## Note

- We really want to know how well the approximations generated by the methods satisfy the differential equation, not the other way around.

# Local Truncation Error

## Informal Definition of LTE

The local truncation error at a specified step measures the amount by which the exact solution to the differential equation fails to satisfy the difference equation being used for the approximation at that step.

## Note

- We really want to know how well the approximations generated by the methods satisfy the differential equation, not the other way around.
- However, we don’t know the exact solution so we cannot generally determine this, and the local truncation will serve quite well to determine not only the local error of a method but the actual approximation error.

# Local Truncation Error

IVP $$y' = f(t,y), \quad a \le t \le b, \quad y(a) = \alpha$$

## Definition of LTE

The difference method

$$w_0 = \alpha$$
$$w_{i+1} = w_i + h\phi(t_i, w_i), \quad \text{for each } i = 0, 1, \ldots, N-1,$$

# Local Truncation Error

IVP $\quad y' = f(t, y), \quad a \le t \le b, \quad y(a) = \alpha$

## Definition of LTE

The difference method

$$w_0 = \alpha$$
$$w_{i+1} = w_i + h\phi(t_i, w_i), \quad \text{for each } i = 0, 1, \ldots, N-1,$$

has local truncation error

$$\tau_{i+1}(h) = \frac{y_{i+1} - (y_i + h\phi(t_i, y_i))}{h} = \frac{y_{i+1} - y_i}{h} - \phi(t_i, y_i),$$

for each $i = 0, 1, \ldots, N-1$, where $y_i$ and $y_{i+1}$ denote the solution at $t_i$ and $t_{i+1}$, respectively.

# Local Truncation Error

## Example: LTE in Euler's Method

Euler's method has local truncation error at the $i$th step

$$\tau_{i+1}(h) = \frac{y_{i+1} - y_i}{h} - f(t_i, y_i), \quad \text{for each } i = 0, 1, \ldots, N-1$$

# Local Truncation Error

## Example: LTE in Euler's Method

Euler's method has local truncation error at the $i$th step

$$\tau_{i+1}(h) = \frac{y_{i+1} - y_i}{h} - f(t_i, y_i), \quad \text{for each } i = 0, 1, \ldots, N-1$$

- This error is a local error because it measures the accuracy of the method at a specific step, assuming that the method was exact at the previous step.

# Local Truncation Error

## Example: LTE in Euler's Method

Euler's method has local truncation error at the $i$th step

$$\tau_{i+1}(h) = \frac{y_{i+1} - y_i}{h} - f(t_i, y_i), \quad \text{for each } i = 0, 1, \ldots, N-1$$

- This error is a local error because it measures the accuracy of the method at a specific step, assuming that the method was exact at the previous step.
- As such, it depends on the differential equation, the step size, and the particular step in the approximation.

# Local Truncation Error

## LTE in Euler's Method (Cont'd)

# Local Truncation Error

## LTE in Euler's Method (Cont'd)

Earlier, we have seen that, for Euler's method:

$$y(t_{i+1}) = y(t_i) + hf(t_i, y(t_i)) + \frac{h^2}{2} y''(\xi_i)$$

# Local Truncation Error

## LTE in Euler's Method (Cont'd)

Earlier, we have seen that, for Euler's method:

$$y(t_{i+1}) = y(t_i) + hf(t_i, y(t_i)) + \frac{h^2}{2}y''(\xi_i)$$

so that the LTE is

$$\tau_{i+1}(h) = \frac{h}{2}y''(\xi_i), \quad \text{for some } \xi_i \text{ in } (t_i, t_{i+1})$$

# Local Truncation Error

## LTE in Euler's Method (Cont'd)

Earlier, we have seen that, for Euler's method:

$$y(t_{i+1}) = y(t_i) + hf(t_i, y(t_i)) + \frac{h^2}{2}y''(\xi_i)$$

so that the LTE is

$$\tau_{i+1}(h) = \frac{h}{2}y''(\xi_i), \quad \text{for some } \xi_i \text{ in } (t_i, t_{i+1})$$

When $y''(t)$ is known to be bounded by a constant $M$ on $[a, b]$, this implies

$$|\tau_{i+1}(h)| \leq \frac{h}{2}M$$

# Local Truncation Error

## LTE in Euler's Method (Cont'd)

Earlier, we have seen that, for Euler's method:

$$y(t_{i+1}) = y(t_i) + hf(t_i, y(t_i)) + \frac{h^2}{2}y''(\xi_i)$$

so that the LTE is

$$\tau_{i+1}(h) = \frac{h}{2}y''(\xi_i), \quad \text{for some } \xi_i \text{ in } (t_i, t_{i+1})$$

When $y''(t)$ is known to be bounded by a constant $M$ on $[a, b]$, this implies

$$|\tau_{i+1}(h)| \leq \frac{h}{2}M$$

so the local truncation error in Euler's method is $O(h)$.

# Outline

# Using Taylor's Theorem

## Motivation

# Using Taylor's Theorem

## Motivation

- One way to select difference-equation methods for solving ordinary differential equations is in such a manner that their local truncation errors are $O(h^p)$ for as large a value of $p$ as possible, $\ldots$

# Using Taylor's Theorem

## Motivation

- One way to select difference-equation methods for solving ordinary differential equations is in such a manner that their local truncation errors are $O(h^p)$ for as large a value of $p$ as possible, . . .
- while keeping the number and complexity of calculations of the methods within a reasonable bound.

# Using Taylor's Theorem

## Motivation

- One way to select difference-equation methods for solving ordinary differential equations is in such a manner that their local truncation errors are $O(h^p)$ for as large a value of $p$ as possible,

  ...
- while keeping the number and complexity of calculations of the methods within a reasonable bound.
- Euler's method was derived by using Taylor's Theorem with $n = 1$ to approximate the solution of the differential equation.

# Using Taylor's Theorem

## Motivation

- One way to select difference-equation methods for solving ordinary differential equations is in such a manner that their local truncation errors are $O(h^p)$ for as large a value of $p$ as possible,

  ...
- while keeping the number and complexity of calculations of the methods within a reasonable bound.
- Euler's method was derived by using Taylor's Theorem with $n = 1$ to approximate the solution of the differential equation.
- Can we extend this technique of derivation to larger values of $n$ in order to find methods for improving the convergence properties of difference methods?

# Higher-Order Taylor Methods

## Assumption

The solution $y(t)$ to the initial-value problem

$$y' = f(t, y), \quad a \leq t \leq b, \quad y(a) = \alpha,$$

has $(n + 1)$ continuous derivatives.

# Higher-Order Taylor Methods

## Assumption

The solution $y(t)$ to the initial-value problem

$$y' = f(t, y), \quad a \leq t \leq b, \quad y(a) = \alpha,$$

has $(n + 1)$ continuous derivatives.

## Taylor Expansion about $t_i$

If we expand the solution, $y(t)$, in terms of its $n$th Taylor polynomial about $t_i$ and evaluate at $t_{i+1}$, we obtain

$$y(t_{i+1}) = y(t_i) + hy'(t_i) + \frac{h^2}{2}y''(t_i) + \cdots + \frac{h^n}{n!}y^{(n)}(t_i) + \frac{h^{n+1}}{(n+1)!}y^{(n+1)}(\xi_i)$$

for some $\xi_i$ in $(t_i, t_{i+1})$.

# Higher-Order Taylor Methods

## Derivation (Cont'd)

# Higher-Order Taylor Methods

## Derivation (Cont'd)

Successive differentiation of the solution, $y(t)$, gives

$$y'(t) = f(t, y(t)), \quad y''(t) = f'(t, y(t)), \quad \ldots \quad y^{(k)}(t) = f^{(k-1)}(t, y(t))$$

# Higher-Order Taylor Methods

## Derivation (Cont'd)

Successive differentiation of the solution, $y(t)$, gives

$$y'(t) = f(t, y(t)), \quad y''(t) = f'(t, y(t)), \quad \ldots \quad y^{(k)}(t) = f^{(k-1)}(t, y(t))$$

Substituting these results into

$$y(t_{i+1}) = y(t_i) + hy'(t_i) + \frac{h^2}{2} y''(t_i) + \cdots + \frac{h^n}{n!} y^{(n)}(t_i) + \frac{h^{n+1}}{(n+1)!} y^{(n+1)}(\xi_i)$$

# Higher-Order Taylor Methods

## Derivation (Cont'd)

Successive differentiation of the solution, $y(t)$, gives

$$y'(t) = f(t, y(t)), \quad y''(t) = f'(t, y(t)), \quad \ldots \quad y^{(k)}(t) = f^{(k-1)}(t, y(t))$$

Substituting these results into

$$y(t_{i+1}) = y(t_i) + hy'(t_i) + \frac{h^2}{2}y''(t_i) + \cdots + \frac{h^n}{n!}y^{(n)}(t_i) + \frac{h^{n+1}}{(n+1)!}y^{(n+1)}(\xi_i)$$

gives

$$\begin{aligned} y(t_{i+1}) &= y(t_i) + hf(t_i, y(t_i)) + \frac{h^2}{2}f'(t_i, y(t_i)) + \cdots \\ &\quad + \frac{h^n}{n!}f^{(n-1)}(t_i, y(t_i)) + \frac{h^{n+1}}{(n+1)!}f^{(n)}(\xi_i, y(\xi_i)) \end{aligned}$$

# Higher-Order Taylor Methods

## Derivation (Cont'd)

# Higher-Order Taylor Methods

## Derivation (Cont'd)

The difference-equation method corresponding to

$$\begin{aligned} y(t_{i+1}) &= y(t_i) + hf(t_i, y(t_i)) + \frac{h^2}{2} f'(t_i, y(t_i)) + \cdots \\ &\quad + \frac{h^n}{n!} f^{(n-1)}(t_i, y(t_i)) + \frac{h^{n+1}}{(n+1)!} f^{(n)}(\xi_i, y(\xi_i)) \end{aligned}$$

is obtained by deleting the remainder term involving $\xi_i$.

# Higher-Order Taylor Methods

## Taylor's Method of order $n$

$$\begin{aligned} w_0 &= \alpha \\ w_{i+1} &= w_i + hT^{(n)}(t_i, w_i), \quad \text{for each } i = 0, 1, \ldots, N-1 \end{aligned}$$

where

$$T^{(n)}(t_i, w_i) = f(t_i, w_i) + \frac{h}{2} f'(t_i, w_i) + \cdots + \frac{h^{n-1}}{n!} f^{(n-1)}(t_i, w_i)$$

Note: Euler's method is Taylor's method of order one.

# Outline

# Higher-Order Taylor Methods

## Example: Orders 2 & 4 Methods

Apply Taylor's method of orders

- 2 and
- 4

with $N = 10$ to the initial-value problem

$$y' = y - t^2 + 1, \quad 0 \leq t \leq 2, \quad y(0) = 0.5$$

# Higher-Order Taylor Methods

## Order 2 Method (1/4)

For the method of order 2 ▸ Taylor's Nethod we need the first derivative of $f(t, y(t)) = y(t) - t^2 + 1$ with respect to the variable $t$.

# Higher-Order Taylor Methods

## Order 2 Method (1/4)

For the method of order 2 ▸ Taylor's Nethod we need the first derivative of $f(t, y(t)) = y(t) - t^2 + 1$ with respect to the variable $t$. Because $y' = y - t^2 + 1$ we have

$$f'(t, y(t)) = \frac{d}{dt}(y - t^2 + 1) = y' - 2t = y - t^2 + 1 - 2t$$

# Higher-Order Taylor Methods

## Order 2 Method (1/4)

For the method of order 2 ▸ Taylor's Nethod we need the first derivative of $f(t, y(t)) = y(t) - t^2 + 1$ with respect to the variable $t$. Because $y' = y - t^2 + 1$ we have

$$f'(t, y(t)) = \frac{d}{dt}(y - t^2 + 1) = y' - 2t = y - t^2 + 1 - 2t$$

so

$$T^{(2)}(t_i, w_i) \quad = \quad f(t_i, w_i) + \frac{h}{2} f'(t_i, w_i)$$

# Higher-Order Taylor Methods

## Order 2 Method (1/4)

For the method of order 2 ▸ Taylor's Nethod we need the first derivative of $f(t, y(t)) = y(t) - t^2 + 1$ with respect to the variable $t$. Because $y' = y - t^2 + 1$ we have

$$f'(t, y(t)) = \frac{d}{dt}(y - t^2 + 1) = y' - 2t = y - t^2 + 1 - 2t$$

so

$$\begin{aligned} T^{(2)}(t_i, w_i) &= f(t_i, w_i) + \frac{h}{2} f'(t_i, w_i) \\ &= w_i - t_i^2 + 1 + \frac{h}{2}(w_i - t_i^2 + 1 - 2t_i) \end{aligned}$$

# Higher-Order Taylor Methods

## Order 2 Method (1/4)

For the method of order 2 ▸ Taylor's Nethod we need the first derivative of $f(t, y(t)) = y(t) - t^2 + 1$ with respect to the variable $t$. Because $y' = y - t^2 + 1$ we have

$$f'(t, y(t)) = \frac{d}{dt}(y - t^2 + 1) = y' - 2t = y - t^2 + 1 - 2t$$

so

$$\begin{aligned} T^{(2)}(t_i, w_i) &= f(t_i, w_i) + \frac{h}{2} f'(t_i, w_i) \\ &= w_i - t_i^2 + 1 + \frac{h}{2}(w_i - t_i^2 + 1 - 2t_i) \\ &= \left(1 + \frac{h}{2}\right)(w_i - t_i^2 + 1) - ht_i \end{aligned}$$

# Higher-Order Taylor Methods

## Order 2 Method (2/4)

Because $N = 10$ we have $h = 0.2$, and $t_i = 0.2i$ for each $i = 1, 2, \ldots, 10$.

# Higher-Order Taylor Methods

## Order 2 Method (2/4)

Because $N = 10$ we have $h = 0.2$, and $t_i = 0.2i$ for each $i = 1, 2, \ldots, 10$. Thus, the second-order method becomes

$$w_0 = 0.5$$

# Higher-Order Taylor Methods

## Order 2 Method (2/4)

Because $N = 10$ we have $h = 0.2$, and $t_i = 0.2i$ for each $i = 1, 2, \ldots, 10$. Thus, the second-order method becomes

$$\begin{aligned} w_0 &= 0.5 \\ w_{i+1} &= w_i + h\left[\left(1 + \frac{h}{2}\right)\left(w_i - t_i^2 + 1\right) - ht_i\right] \end{aligned}$$

# Higher-Order Taylor Methods

## Order 2 Method (2/4)

Because $N = 10$ we have $h = 0.2$, and $t_i = 0.2i$ for each $i = 1, 2, \ldots, 10$. Thus, the second-order method becomes

$$\begin{aligned} w_0 &= 0.5 \\ w_{i+1} &= w_i + h\left[\left(1 + \frac{h}{2}\right)\left(w_i - t_i^2 + 1\right) - ht_i\right] \\ &= w_i + 0.2\left[\left(1 + \frac{0.2}{2}\right)(w_i - 0.04i^2 + 1) - 0.04i\right] \end{aligned}$$

# Higher-Order Taylor Methods

## Order 2 Method (2/4)

Because $N = 10$ we have $h = 0.2$, and $t_i = 0.2i$ for each $i = 1, 2, \ldots, 10$. Thus, the second-order method becomes

$$\begin{aligned} w_0 &= 0.5 \\ w_{i+1} &= w_i + h\left[\left(1 + \frac{h}{2}\right)\left(w_i - t_i^2 + 1\right) - ht_i\right] \\ &= w_i + 0.2\left[\left(1 + \frac{0.2}{2}\right)(w_i - 0.04i^2 + 1) - 0.04i\right] \\ &= 1.22w_i - 0.0088i^2 - 0.008i + 0.22 \end{aligned}$$

# Higher-Order Taylor Methods

## Order 2 Method (3/4)

# Higher-Order Taylor Methods

## Order 2 Method (3/4)

The first two steps give the approximations

$$\begin{aligned} y(0.2) \approx w_1 &= 1.22(0.5) - 0.0088(0)^2 - 0.008(0) + 0.22 \\ &= 0.83 \end{aligned}$$

# Higher-Order Taylor Methods

## Order 2 Method (3/4)

The first two steps give the approximations

$$\begin{aligned} y(0.2) \approx w_1 &= 1.22(0.5) - 0.0088(0)^2 - 0.008(0) + 0.22 \\ &= 0.83 \\ y(0.4) \approx w_2 &= 1.22(0.83) - 0.0088(0.2)^2 - 0.008(0.2) + 0.22 \\ &= 1.2158 \end{aligned}$$

# Higher-Order Taylor Methods

## Order 2 Method (3/4)

The first two steps give the approximations

$$\begin{aligned} y(0.2) \approx w_1 &= 1.22(0.5) - 0.0088(0)^2 - 0.008(0) + 0.22 \\ &= 0.83 \\ y(0.4) \approx w_2 &= 1.22(0.83) - 0.0088(0.2)^2 - 0.008(0.2) + 0.22 \\ &= 1.2158 \end{aligned}$$

All the approximations and their errors are shown in the following table.

# Higher-Order Taylor Methods

## Order 2 Method (4/4): Summary of Numerical Results

| $t_i$ | Taylor Order 2 $w_i$ | Error $\lvert y(t_i) - w_i \rvert$ |
|---|---|---|
| 0.0 | 0.500000 | 0 |
| 0.2 | 0.830000 | 0.000701 |
| 0.4 | 1.215800 | 0.001712 |
| 0.6 | 1.652076 | 0.003135 |
| 0.8 | 2.132333 | 0.005103 |
| ⋮ | ⋮ | ⋮ |
| 1.6 | 4.306146 | 0.022663 |
| 1.8 | 4.846299 | 0.031122 |
| 2.0 | 5.347684 | 0.042212 |

# Higher-Order Taylor Methods

## Order 4 Method (1/7)

For the method of order 4 ▸ Taylor's Nethod we need the first 3 derivatives of $f(t, y(t))$ with respect to $t$.

# Higher-Order Taylor Methods

## Order 4 Method (1/7)

For the method of order 4 ▸ Taylor's Nethod we need the first 3 derivatives of $f(t, y(t))$ with respect to $t$. Again using $y' = y - t^2 + 1$ we have

$$f'(t, y(t)) = y - t^2 + 1 - 2t$$

# Higher-Order Taylor Methods

## Order 4 Method (1/7)

For the method of order 4 Taylor's Nethod we need the first 3 derivatives of $f(t, y(t))$ with respect to $t$. Again using $y' = y - t^2 + 1$ we have

$$\begin{aligned} f'(t, y(t)) &= y - t^2 + 1 - 2t \\ f''(t, y(t)) &= \frac{d}{dt}(y - t^2 + 1 - 2t) \end{aligned}$$

# Higher-Order Taylor Methods

## Order 4 Method (1/7)

For the method of order 4 ▸ Taylor's Nethod we need the first 3 derivatives of $f(t, y(t))$ with respect to $t$. Again using $y' = y - t^2 + 1$ we have

$$
\begin{aligned}
f'(t, y(t)) &= y - t^2 + 1 - 2t \\
f''(t, y(t)) &= \frac{d}{dt}(y - t^2 + 1 - 2t) = y' - 2t - 2
\end{aligned}
$$

# Higher-Order Taylor Methods

## Order 4 Method (1/7)

For the method of order 4 ▸ Taylor's Nethod we need the first 3 derivatives of $f(t, y(t))$ with respect to $t$. Again using $y' = y - t^2 + 1$ we have

$$
\begin{aligned}
f'(t, y(t)) &= y - t^2 + 1 - 2t \\
f''(t, y(t)) &= \frac{d}{dt}(y - t^2 + 1 - 2t) = y' - 2t - 2 \\
&= y - t^2 + 1 - 2t - 2
\end{aligned}
$$

# Higher-Order Taylor Methods

## Order 4 Method (1/7)

For the method of order 4 Taylor's Nethod we need the first 3 derivatives of $f(t, y(t))$ with respect to $t$. Again using $y' = y - t^2 + 1$ we have

$$
\begin{aligned}
f'(t, y(t)) &= y - t^2 + 1 - 2t \\
f''(t, y(t)) &= \frac{d}{dt}(y - t^2 + 1 - 2t) = y' - 2t - 2 \\
&= y - t^2 + 1 - 2t - 2 = y - t^2 - 2t - 1
\end{aligned}
$$

# Higher-Order Taylor Methods

## Order 4 Method (1/7)

For the method of order 4 ▸ Taylor's Nethod we need the first 3 derivatives of $f(t, y(t))$ with respect to $t$. Again using $y' = y - t^2 + 1$ we have

$$\begin{aligned} f'(t, y(t)) &= y - t^2 + 1 - 2t \\ f''(t, y(t)) &= \frac{d}{dt}(y - t^2 + 1 - 2t) = y' - 2t - 2 \\ &= y - t^2 + 1 - 2t - 2 = y - t^2 - 2t - 1 \\ \text{and} \quad f'''(t, y(t)) &= \frac{d}{dt}(y - t^2 - 2t - 1) \end{aligned}$$

# Higher-Order Taylor Methods

## Order 4 Method (1/7)

For the method of order 4 Taylor's Nethod we need the first 3 derivatives of $f(t, y(t))$ with respect to $t$. Again using $y' = y - t^2 + 1$ we have

$$
\begin{aligned}
f'(t, y(t)) &= y - t^2 + 1 - 2t \\
f''(t, y(t)) &= \frac{d}{dt}(y - t^2 + 1 - 2t) = y' - 2t - 2 \\
&= y - t^2 + 1 - 2t - 2 = y - t^2 - 2t - 1 \\
\text{and } f'''(t, y(t)) &= \frac{d}{dt}(y - t^2 - 2t - 1) = y' - 2t - 2
\end{aligned}
$$

# Higher-Order Taylor Methods

## Order 4 Method (1/7)

For the method of order 4 ▸ Taylor's Nethod we need the first 3 derivatives of $f(t, y(t))$ with respect to $t$. Again using $y' = y - t^2 + 1$ we have

$$
\begin{aligned}
f'(t, y(t)) &= y - t^2 + 1 - 2t \\
f''(t, y(t)) &= \frac{d}{dt}(y - t^2 + 1 - 2t) = y' - 2t - 2 \\
&= y - t^2 + 1 - 2t - 2 = y - t^2 - 2t - 1 \\
\text{and} \quad f'''(t, y(t)) &= \frac{d}{dt}(y - t^2 - 2t - 1) = y' - 2t - 2 \\
&= y - t^2 - 2t - 1
\end{aligned}
$$

# Higher-Order Taylor Methods

## Order 4 Method (2/7)

# Higher-Order Taylor Methods

## Order 4 Method (2/7)

Therefore,

$$T^{(4)}(t_i, w_i) = f(t_i, w_i) + \frac{h}{2} f'(t_i, w_i) + \frac{h^2}{6} f''(t_i, w_i) + \frac{h^3}{24} f'''(t_i, w_i)$$

# Higher-Order Taylor Methods

## Order 4 Method (2/7)

Therefore,

$$\begin{aligned} T^{(4)}(t_i, w_i) &= f(t_i, w_i) + \frac{h}{2} f'(t_i, w_i) + \frac{h^2}{6} f''(t_i, w_i) + \frac{h^3}{24} f'''(t_i, w_i) \\ &= w_i - t_i^2 + 1 + \frac{h}{2}(w_i - t_i^2 + 1 - 2t_i) \\ &\quad + \frac{h^2}{6}(w_i - t_i^2 - 2t_i - 1) + \frac{h^3}{24}(w_i - t_i^2 - 2t_i - 1) \end{aligned}$$

# Higher-Order Taylor Methods

## Order 4 Method (2/7)

Therefore,

$$
\begin{aligned}
T^{(4)}(t_i, w_i) &= f(t_i, w_i) + \frac{h}{2} f'(t_i, w_i) + \frac{h^2}{6} f''(t_i, w_i) + \frac{h^3}{24} f'''(t_i, w_i) \\
&= w_i - t_i^2 + 1 + \frac{h}{2}(w_i - t_i^2 + 1 - 2t_i) \\
&\quad + \frac{h^2}{6}(w_i - t_i^2 - 2t_i - 1) + \frac{h^3}{24}(w_i - t_i^2 - 2t_i - 1) \\
&= \left(1 + \frac{h}{2} + \frac{h^2}{6} + \frac{h^3}{24}\right)(w_i - t_i^2) - \left(1 + \frac{h}{3} + \frac{h^2}{12}\right)(ht_i) \\
&\quad + 1 + \frac{h}{2} - \frac{h^2}{6} - \frac{h^3}{24}
\end{aligned}
$$

# Higher-Order Taylor Methods

## Order 4 Method (3/7)

# Higher-Order Taylor Methods

## Order 4 Method (3/7)

Hence Taylor's method of order four is

$$\begin{aligned} w_0 &= 0.5, \\ w_{i+1} &= w_i + h\left[\left(1 + \frac{h}{2} + \frac{h^2}{6} + \frac{h^3}{24}\right)(w_i - t_i^2) - \left(1 + \frac{h}{3} + \frac{h^2}{12}\right)ht_i \right. \\ &\quad \left. + 1 + \frac{h}{2} - \frac{h^2}{6} - \frac{h^3}{24}\right] \end{aligned}$$

for $i = 0, 1, \ldots, N-1$.

# Higher-Order Taylor Methods

## Order 4 Method (4/7)

Because $N = 10$ and $h = 0.2$,

# Higher-Order Taylor Methods

## Order 4 Method (4/7)

Because $N = 10$ and $h = 0.2$, the method becomes

$$\begin{aligned} w_{i+1} &= w_i + 0.2\left[\left(1 + \frac{0.2}{2} + \frac{0.04}{6} + \frac{0.008}{24}\right)(w_i - 0.04i^2)\right. \\ &\quad \left. - \left(1 + \frac{0.2}{3} + \frac{0.04}{12}\right)(0.04i) + 1 + \frac{0.2}{2} - \frac{0.04}{6} - \frac{0.008}{24}\right] \end{aligned}$$

# Higher-Order Taylor Methods

## Order 4 Method (4/7)

Because $N = 10$ and $h = 0.2$, the method becomes

$$\begin{aligned} w_{i+1} &= w_i + 0.2\left[\left(1 + \frac{0.2}{2} + \frac{0.04}{6} + \frac{0.008}{24}\right)(w_i - 0.04i^2)\right. \\ &\quad \left. - \left(1 + \frac{0.2}{3} + \frac{0.04}{12}\right)(0.04i) + 1 + \frac{0.2}{2} - \frac{0.04}{6} - \frac{0.008}{24}\right] \\ &= 1.2214w_i - 0.008856i^2 - 0.00856i + 0.2186 \end{aligned}$$

for each $i = 0, 1, \ldots, 9$.

# Higher-Order Taylor Methods

## Order 4 Method (5/7)

# Higher-Order Taylor Methods

## Order 4 Method (5/7)

The first two steps give the approximations

$$\begin{aligned} y(0.2) \approx w_1 &= 1.2214(0.5) - 0.008856(0)^2 - 0.00856(0) + 0.2186 \\ &= 0.8293 \end{aligned}$$

# Higher-Order Taylor Methods

## Order 4 Method (5/7)

The first two steps give the approximations

$$
\begin{aligned}
y(0.2) \approx w_1 &= 1.2214(0.5) - 0.008856(0)^2 - 0.00856(0) + 0.2186 \\
&= 0.8293 \\
y(0.4) \approx w_2 &= 1.2214(0.8293) - 0.008856(0.2)^2 - 0.00856(0.2) \\
&\qquad +0.2186 \\
&= 1.214091
\end{aligned}
$$

# Higher-Order Taylor Methods

## Order 4 Method (5/7)

The first two steps give the approximations

$$
\begin{aligned}
y(0.2) \approx w_1 &= 1.2214(0.5) - 0.008856(0)^2 - 0.00856(0) + 0.2186 \\
&= 0.8293 \\
y(0.4) \approx w_2 &= 1.2214(0.8293) - 0.008856(0.2)^2 - 0.00856(0.2) \\
&\qquad + 0.2186 \\
&= 1.214091
\end{aligned}
$$

All the approximations and their errors are shown in the following table.

# Higher-Order Taylor Methods

## Order 4 Method (6/7): Summary of Numerical Results

| $t_i$ | Taylor Order 4 $w_i$ | Error $\lvert y(t_i) - w_i \rvert$ |
|---|---|---|
| 0.0 | 0.500000 | 0 |
| 0.2 | 0.829300 | 0.000001 |
| 0.4 | 1.214091 | 0.000003 |
| 0.6 | 1.648947 | 0.000006 |
| $\vdots$ | $\vdots$ | $\vdots$ |
| 1.4 | 3.732432 | 0.000032 |
| 1.6 | 4.283529 | 0.000045 |
| 1.8 | 4.815238 | 0.000062 |
| 2.0 | 5.305555 | 0.000083 |

# Higher-Order Taylor Methods

## Order 4 Method (7/7)

# Higher-Order Taylor Methods

## Order 4 Method (7/7)

- A comparison of these results with those of Taylor's method of order 2 shows that the 4th-order results are vastly superior.

# Higher-Order Taylor Methods

## Order 4 Method (7/7)

- A comparison of these results with those of Taylor's method of order 2 shows that the 4th-order results are vastly superior.
- The table of results for Taylor's method of order 4 indicate that the method is quite accurate at the nodes 0.2, 0.4, etc.

# Outline

# Higher-Order Taylor Methods

## Theorem

If Taylor's method of order $n$ is used to approximate the solution to

$$y'(t) = f(t, y(t)), \quad a \leq t \leq b, \quad y(a) = \alpha,$$

with step size $h$ and if $y \in C^{n+1}[a, b]$, then the local truncation error is $O(h^n)$.

# Higher-Order Taylor Methods

## Proof (1/2)

# Higher-Order Taylor Methods

## Proof (1/2)

When deriving Taylor Methods, we obtained the expression

$$y(t_{i+1}) = y(t_i) + hy'(t_i) + \frac{h^2}{2}y''(t_i) + \cdots + \frac{h^n}{n!}y^{(n)}(t_i) + \frac{h^{n+1}}{(n+1)!}y^{(n+1)}(\xi_i)$$

# Higher-Order Taylor Methods

## Proof (1/2)

When deriving Taylor Methods, we obtained the expression

$$y(t_{i+1}) = y(t_i) + hy'(t_i) + \frac{h^2}{2}y''(t_i) + \cdots + \frac{h^n}{n!}y^{(n)}(t_i) + \frac{h^{n+1}}{(n+1)!}y^{(n+1)}(\xi_i)$$

and this can be rewritten in the form

$$\begin{aligned} & y_{i+1} - y_i - hf(t_i, y_i) - \frac{h^2}{2}f'(t_i, y_i) - \cdots - \frac{h^n}{n!}f^{(n-1)}(t_i, y_i) \\ = \; & \frac{h^{n+1}}{(n+1)!}f^{(n)}(\xi_i, y(\xi_i)) \end{aligned}$$

for some $\xi_i$ in $(t_i, t_{i+1})$.

# Higher-Order Taylor Methods

$$y_{i+1} - y_i - hf(t_i, y_i) - \frac{h^2}{2} f'(t_i, y_i) - \cdots - \frac{h^n}{n!} f^{(n-1)}(t_i, y_i)$$
$$= \frac{h^{n+1}}{(n+1)!} f^{(n)}(\xi_i, y(\xi_i))$$

## Proof (2/2)

# Higher-Order Taylor Methods

$$y_{i+1} - y_i - hf(t_i, y_i) - \frac{h^2}{2} f'(t_i, y_i) - \cdots - \frac{h^n}{n!} f^{(n-1)}(t_i, y_i)$$
$$= \frac{h^{n+1}}{(n+1)!} f^{(n)}(\xi_i, y(\xi_i))$$

## Proof (2/2)

So the local truncation error is

$$\tau_{i+1}(h) = \frac{y_{i+1} - y_i}{h} - T^{(n)}(t_i, y_i) = \frac{h^n}{(n+1)!} f^{(n)}(\xi_i, y(\xi_i))$$

for each $i = 0, 1, \ldots, N-1$.

# Higher-Order Taylor Methods

$$y_{i+1} - y_i - hf(t_i, y_i) - \frac{h^2}{2} f'(t_i, y_i) - \cdots - \frac{h^n}{n!} f^{(n-1)}(t_i, y_i)$$
$$= \frac{h^{n+1}}{(n+1)!} f^{(n)}(\xi_i, y(\xi_i))$$

## Proof (2/2)

So the local truncation error is

$$\tau_{i+1}(h) = \frac{y_{i+1} - y_i}{h} - T^{(n)}(t_i, y_i) = \frac{h^n}{(n+1)!} f^{(n)}(\xi_i, y(\xi_i))$$

for each $i = 0, 1, \ldots, N-1$. Since $y \in C^{n+1}[a, b]$, we have $y^{(n+1)}(t) = f^{(n)}(t, y(t))$ bounded on $[a, b]$ and $\tau_i(h) = O(h^n)$, for each $i = 1, 2, \ldots, N$.

Questions?

# Reference Material

# Higher-Order Taylor Methods

## Taylor's Method of order $n$

$$\begin{aligned} w_0 &= \alpha \\ w_{i+1} &= w_i + hT^{(n)}(t_i, w_i), \quad \text{for each } i = 0, 1, \ldots, N-1 \end{aligned}$$

where

$$T^{(n)}(t_i, w_i) = f(t_i, w_i) + \frac{h}{2} f'(t_i, w_i) + \cdots + \frac{h^{n-1}}{n!} f^{(n-1)}(t_i, w_i)$$

◂ Return to Example on Taylor's 2nd Order Method

◂ Return to Example on Taylor's 4th Order Method