Initial-Value Problems for ODEs

Runge-Kutta Methods

Numerical Analysis (9th Edition)

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Beamer Presentation Slides
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Outline

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1. Introduction & Taylor's Theorem in 2 Variables
2. Runge-Kutta Methods of Order Two

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Runge-Kutta Methods

Taylor Methods .v. Runge-Kutta Methods

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- Taylor methods have the desirable property of high-order local truncation error,
- but the disadvantage of requiring the computation and evaluation of the derivatives of $f(t, y)$.
- This is a complicated and time-consuming procedure for most problems, so the Taylor methods are seldom used in practice.
- Runge-Kutta methods have the high-order local truncation error of the Taylor methods but eliminate the need to compute and evaluate the derivatives of $f(t, y)$.

Taylor Theorem in 2 Variables (1/2)

Suppose that $f(t, y)$ and all its partial derivatives of order less than or equal to $n + 1$ are continuous on $D = \{ (t, y) \mid a \leq t \leq b, c \leq y \leq d \}$, and let $(t_0, y_0) \in D$. For every $(t, y) \in D$, there exists ξ between t and t_0 and μ between y and y_0 with

$$f(t, y) = P_n(t, y) + R_n(t, y)$$

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$$f(t, y) = P_n(t, y) + R_n(t, y)$$

The function $P_n(t, y)$ is called the ***n*th Taylor polynomial in two variables** for the function f about (t_0, y_0), and $R_n(t, y)$ is the remainder term associated with $P_n(t, y)$.

Details of $P_n(t, y)$ and $R_n(t, y)$ are given on the next slide.

Taylor Theorem in 2 Variables (2/2)

$$
\begin{aligned}
P_n(t,y) \;=\;& f(t_0,y_0) + \left[(t-t_0)\frac{\partial f}{\partial t}(t_0,y_0) + (y-y_0)\frac{\partial f}{\partial y}(t_0,y_0)\right] \\
&+ \left[\frac{(t-t_0)^2}{2}\frac{\partial^2 f}{\partial t^2}(t_0,y_0) + (t-t_0)(y-y_0)\frac{\partial^2 f}{\partial t \partial y}(t_0,y_0)\right. \\
&+ \left.\frac{(y-y_0)^2}{2}\frac{\partial^2 f}{\partial y^2}(t_0,y_0)\right] + \cdots \\
&+ \left[\frac{1}{n!}\sum_{j=0}^{n}\binom{n}{j}(t-t_0)^{n-j}(y-y_0)^j \frac{\partial^n f}{\partial t^{n-j}\partial y^j}(t_0,y_0)\right]
\end{aligned}
$$

$$
R_n(t,y) = \frac{1}{(n+1)!}\sum_{j=0}^{n+1}\binom{n+1}{j}(t-t_0)^{n+1-j}(y-y_0)^j \frac{\partial^{n+1} f}{\partial t^{n+1-j}\partial y^j}(\xi,\mu)
$$

Outline

2nd Order Runge-Kutta Methods

Basic Structure of RK2 Methods

Our starting point is to assume that the numerical method has the following structure:

$$\begin{aligned} w_0 &= \alpha \\ w_{i+1} &= w_i + a_1 f\left(t_i + \alpha_1, w_i + \beta_1 f\left(t_i, w_i\right)\right) \end{aligned}$$

for $i = 0, 1, \ldots, N - 1$, where a_1, α_1 and β_1 are parameters to be determined to ensure a local truncation error of $O(h^2)$.

2nd Order Runge-Kutta Methods

Method of Derivation

2nd Order Runge-Kutta Methods

Method of Derivation

The first step is to determine values for a_1, α_1, and β_1 with the property that

$$a_1 f(t+\alpha_1, y+\beta_1) \approx T^{(2)}(t,y) = f(t,y) + \frac{h}{2} f'(t,y),$$

with error no greater than $O(h^2)$, which is same as the order of the local truncation error for the Taylor method of order two.

2nd Order Runge-Kutta Methods

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with error no greater than $O(h^2)$, which is same as the order of the local truncation error for the Taylor method of order two. Since

$$f^{\prime}(t, y)=\frac{d f}{d t}(t, y)=\frac{\partial f}{\partial t}(t, y)+\frac{\partial f}{\partial y}(t, y) \cdot y^{\prime}(t) \quad \text{and} \quad y^{\prime}(t)=f(t, y),$$

2nd Order Runge-Kutta Methods

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$$a_1 f(t+\alpha_1, y+\beta_1) \approx T^{(2)}(t,y) = f(t,y) + \frac{h}{2}f'(t,y),$$

with error no greater than $O(h^2)$, which is same as the order of the local truncation error for the Taylor method of order two. Since

$$f'(t,y) = \frac{df}{dt}(t,y) = \frac{\partial f}{\partial t}(t,y) + \frac{\partial f}{\partial y}(t,y)\cdot y'(t) \quad \text{and} \quad y'(t) = f(t,y),$$

we have

$$T^{(2)}(t,y) = f(t,y) + \frac{h}{2}\frac{\partial f}{\partial t}(t,y) + \frac{h}{2}\frac{\partial f}{\partial y}(t,y)\cdot f(t,y)$$

2nd Order Runge-Kutta Methods

Method of Derivation (Cont'd)

Expanding $f(t+\alpha_1, y+\beta_1)$ in its Taylor polynomial of degree one about (t, y) gives

$$\begin{aligned} a_1 f(t+\alpha_1, y+\beta_1) &= a_1 f(t,y) + a_1\alpha_1 \frac{\partial f}{\partial t}(t,y) \\ &\quad + a_1\beta_1 \frac{\partial f}{\partial y}(t,y) + a_1 \cdot R_1(t+\alpha_1, y+\beta_1) \end{aligned}$$

2nd Order Runge-Kutta Methods

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where

$$R_1(t+\alpha_1, y+\beta_1) = \frac{\alpha_1^2}{2}\frac{\partial^2 f}{\partial t^2}(\xi,\mu) + \alpha_1\beta_1 \frac{\partial^2 f}{\partial t \partial y}(\xi,\mu) + \frac{\beta_1^2}{2}\frac{\partial^2 f}{\partial y^2}(\xi,\mu)$$

for some ξ between t and $t+\alpha_1$ and μ between y and $y+\beta_1$.

2nd Order Runge-Kutta Methods

Method of Derivation (Cont'd)

Matching the coefficients of f and its derivatives in

$$\begin{aligned} a_1 f(t+\alpha_1, y+\beta_1) &= a_1 f(t,y) + a_1\alpha_1 \frac{\partial f}{\partial t}(t,y) \\ &\quad + a_1\beta_1 \frac{\partial f}{\partial y}(t,y) + a_1 \cdot R_1(t+\alpha_1, y+\beta_1) \end{aligned}$$

and

$$T^{(2)}(t,y) = f(t,y) + \frac{h}{2}\frac{\partial f}{\partial t}(t,y) + \frac{h}{2}\frac{\partial f}{\partial y}(t,y)\cdot f(t,y)$$

2nd Order Runge-Kutta Methods

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and

$$T^{(2)}(t,y) = f(t,y) + \frac{h}{2}\frac{\partial f}{\partial t}(t,y) + \frac{h}{2}\frac{\partial f}{\partial y}(t,y) \cdot f(t,y)$$

gives the three equations

$$a_1 = 1 \qquad a_1\alpha_1 = \frac{h}{2} \qquad a_1\beta_1 = \frac{h}{2} f(t,y)$$

2nd Order Runge-Kutta Methods

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2nd Order Runge-Kutta Methods

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Method of Derivation (Cont'd)

The parameters a_1, α_1, and β_1 are therefore

$$a_1 = 1 \qquad \alpha_1 = \frac{h}{2} \qquad \beta_1 = \frac{h}{2}f(t,y)$$

so that

$$T^{(2)}(t,y) = f\left(t + \frac{h}{2}, y + \frac{h}{2}f(t,y)\right) - R_1\left(t + \frac{h}{2}, y + \frac{h}{2}f(t,y)\right)$$

2nd Order Runge-Kutta Methods

Method of Derivation (Cont'd)

Earlier, we saw that

$$R_1(t+\alpha_1, y+\beta_1) = \frac{\alpha_1^2}{2}\frac{\partial^2 f}{\partial t^2}(\xi,\mu) + \alpha_1\beta_1\frac{\partial^2 f}{\partial t\partial y}(\xi,\mu) + \frac{\beta_1^2}{2}\frac{\partial^2 f}{\partial y^2}(\xi,\mu)$$

2nd Order Runge-Kutta Methods

Method of Derivation (Cont'd)

Earlier, we saw that

$$R_1(t+\alpha_1, y+\beta_1) = \frac{\alpha_1^2}{2}\frac{\partial^2 f}{\partial t^2}(\xi,\mu) + \alpha_1\beta_1\frac{\partial^2 f}{\partial t \partial y}(\xi,\mu) + \frac{\beta_1^2}{2}\frac{\partial^2 f}{\partial y^2}(\xi,\mu)$$

which leads to

$$\begin{aligned} R_1\left(t+\frac{h}{2}, y+\frac{h}{2}f(t,y)\right) &= \frac{h^2}{8}\frac{\partial^2 f}{\partial t^2}(\xi,\mu) + \frac{h^2}{4}f(t,y)\frac{\partial^2 f}{\partial t \partial y}(\xi,\mu) \\ &\quad + \frac{h^2}{8}(f(t,y))^2\frac{\partial^2 f}{\partial y^2}(\xi,\mu). \end{aligned}$$

which is $O(h^2)$ if all the second-order partial derivatives of f are bounded.

2nd Order Runge-Kutta Methods

The difference-equation method resulting from replacing $T^{(2)}(t, y)$ in Taylor's method of order two by $f(t + (h/2), y + (h/2)f(t, y))$ is a specific Runge-Kutta method known as the Midpoint Method.

2nd Order Runge-Kutta Methods

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The Midpoint Method

$$
\begin{aligned}
w_0 &= \alpha, \\
w_{i+1} &= w_i + hf\left(t_i + \frac{h}{2}, w_i + \frac{h}{2}f(t_i, w_i)\right)
\end{aligned}
$$

for $i = 0, 1, \ldots, N - 1$.

2nd Order Runge-Kutta Methods

Number of Parameters Required

- Only three parameters are present in

$$a_1 f\left(t_i + \alpha_1, w_i + \beta_1 f\left(t_i, w_i\right)\right)$$

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- So a more complicated form is required to satisfy the conditions for any of the higher-order Taylor methods.

2nd Order Runge-Kutta Methods

Number of Parameters Required (Cont'd)

The most appropriate four-parameter form for approximating

$$T^{(3)}(t,y) = f(t,y) + \frac{h}{2}f'(t,y) + \frac{h^2}{6}f''(t,y)$$

2nd Order Runge-Kutta Methods

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$$T^{(3)}(t,y) = f(t,y) + \frac{h}{2}f'(t,y) + \frac{h^2}{6}f''(t,y)$$

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$$a_1 f(t,y) + a_2 f(t + \alpha_2, y + \delta_2 f(t,y))$$

2nd Order Runge-Kutta Methods

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is

$$a_1 f(t,y) + a_2 f(t+\alpha_2, y+\delta_2 f(t,y))$$

and even with this, there is insufficient flexibility to match the term

$$\frac{h^2}{6}\left[\frac{\partial f}{\partial y}(t,y)\right]^2 f(t,y),$$

resulting from the expansion of $(h^2/6)f''(t,y)$.

2nd Order Runge-Kutta Methods

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and even with this, there is insufficient flexibility to match the term

$$\frac{h^2}{6}\left[\frac{\partial f}{\partial y}(t,y)\right]^2 f(t,y),$$

resulting from the expansion of $(h^2/6)f''(t,y)$. Consequently, the best that can be obtained from using this form are methods with $O(h^2)$ local truncation error.

2nd Order Runge-Kutta Methods

The fact that

$$a_1 f(t,y) + a_2 f(t+\alpha_2, y+\delta_2 f(t,y))$$

has four parameters, however, gives a flexibility in their choice, so a number of $O(h^2)$ methods can be derived. One of the most important is the Modified Euler method, which corresponds to choosing $a_1 = a_2 = \frac{1}{2}$ and $\alpha_2 = \delta_2 = h$.

Modified Euler Method

$$\begin{aligned} w_0 &= \alpha \\ w_{i+1} &= w_i + \frac{h}{2}\left[f\left(t_i, w_i\right) + f\left(t_{i+1}, w_i + hf\left(t_i, w_i\right)\right)\right] \end{aligned}$$

for $i = 0, 1, \ldots, N-1$.

Outline

Comparing 2nd Order Runge-Kutta Methods

Example

Use the Midpoint Method and the Modified Euler Method with $N = 10$, $h = 0.2$, $t_i = 0.2i$, and $w_0 = 0.5$ to approximate the solution to our usual example,

$$y' = y - t^2 + 1, \quad 0 \leq t \leq 2, \quad y(0) = 0.5.$$

Comparing 2nd Order Runge-Kutta Methods

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$$y' = y - t^2 + 1, \quad 0 \le t \le 2, \quad y(0) = 0.5.$$

Note

The difference equations produced from the two formulae are

$$\begin{aligned} \text{Midpoint:} \quad & w_{i+1} = 1.22w_i - 0.0088i^2 - 0.008i + 0.218 \\ \text{Modified Euler:} \quad & w_{i+1} = 1.22w_i - 0.0088i^2 - 0.008i + 0.216 \end{aligned}$$

for each $i = 0, 1, \ldots, 9$.

Comparing 2nd Order Runge-Kutta Methods

Solution (1/2): Computing the first 2 steps

The first two steps of the Midpoint method give:

$$w_1 = 1.22(0.5) - 0.0088(0)^2 - 0.008(0) + 0.218 = 0.828$$

Comparing 2nd Order Runge-Kutta Methods

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$$\begin{aligned} w_1 &= 1.22(0.5) - 0.0088(0)^2 - 0.008(0) + 0.218 = 0.828 \\ w_2 &= 1.22(0.828) - 0.0088(0.2)^2 - 0.008(0.2) + 0.218 = 1.21136 \end{aligned}$$

Comparing 2nd Order Runge-Kutta Methods

Solution (1/2): Computing the first 2 steps

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while the first two steps of the Modified Euler method give:

$$w_1 = 1.22(0.5) - 0.0088(0)^2 - 0.008(0) + 0.216 = 0.826$$

Comparing 2nd Order Runge-Kutta Methods

Solution (1/2): Computing the first 2 steps

The first two steps of the Midpoint method give:

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while the first two steps of the Modified Euler method give:

$$\begin{aligned} w_1 &= 1.22(0.5) - 0.0088(0)^2 - 0.008(0) + 0.216 = 0.826 \\ w_2 &= 1.22(0.826) - 0.0088(0.2)^2 - 0.008(0.2) + 0.216 = 1.20692 \end{aligned}$$

Comparing 2nd Order Runge-Kutta Methods

Solution (2/2): Tabulated Results for both methods

t_i	$y(t_i)$	Midpoint Method	Error	Modified Euler Method	Error
0.0	0.5000000	0.5000000	0	0.5000000	0
0.2	0.8292986	0.8280000	0.0013	0.8260000	0.0033
0.4	1.2140877	1.2113600	0.0027	1.2069200	0.0072
0.6	1.6489406	1.6446592	0.0043	1.6372424	0.0117
0.8	2.1272295	2.1212842	0.0059	2.1102357	0.0170
1.0	2.6408591	2.6331668	0.0077	2.6176876	0.0232
1.2	3.1799415	3.1704634	0.0095	3.1495789	0.0304
1.4	3.7324000	3.7211654	0.0112	3.6936862	0.0387
1.6	4.2834838	4.2706218	0.0129	4.2350972	0.0484
1.8	4.8151763	4.8009586	0.0142	4.7556185	0.0596
2.0	5.3054720	5.2903695	0.0151	5.2330546	0.0724

Outline

Higher-Order Runge-Kutta Methods

The Heun Method of order 3

Higher-Order Runge-Kutta Methods

The Heun Method of order 3

The term $T^{(3)}(t, y)$ can be approximated with error $O(h^3)$ by an expression of the form

$$f(t + \alpha_1, y + \delta_1 f(t + \alpha_2, y + \delta_2 f(t, y)))$$

involving 4 parameters,

Higher-Order Runge-Kutta Methods

The Heun Method of order 3

The term $T^{(3)}(t, y)$ can be approximated with error $O(h^3)$ by an expression of the form

$$f(t + \alpha_1, y + \delta_1 f(t + \alpha_2, y + \delta_2 f(t, y)))$$

involving 4 parameters, but the algebra involved in the determination of $\alpha_1, \delta_1, \alpha_2$, and δ_2 is quite involved.

Higher-Order Runge-Kutta Methods

The Heun Method of order 3

The term $T^{(3)}(t, y)$ can be approximated with error $O(h^3)$ by an expression of the form

$$f(t + \alpha_1, y + \delta_1 f(t + \alpha_2, y + \delta_2 f(t, y)))$$

involving 4 parameters, but the algebra involved in the determination of $\alpha_1, \delta_1, \alpha_2$, and δ_2 is quite involved. The most common $O(h^3)$ method is that of Heun, given by

$$\begin{aligned} w_0 &= \alpha \\ w_{i+1} &= w_i + \frac{h}{4}\left(f(t_i, w_i)\right. \\ &\qquad \left. +3\left(f\left(t_i + \frac{2h}{3}, w_i + \frac{2h}{3} f\left(t_i + \frac{h}{3}, w_i + \frac{h}{3} f(t_i, w_i)\right)\right)\right)\right) \end{aligned}$$

for $i = 0, 1, \ldots, N - 1$.

Higher-Order Runge-Kutta Methods

Example: The Heun Method

Applying Heun's method with $N = 10$, $h = 0.2$, $t_i = 0.2i$, and $w_0 = 0.5$ to approximate the solution to the equation:

$$y' = y - t^2 + 1, \quad 0 \leq t \leq 2, \quad y(0) = 0.5.$$

gives the values listed in the following table.

Note the decreased error throughout the range over the Midpoint and Modified Euler approximations.

Higher-Order Runge-Kutta Methods

t_i	$y(t_i)$	Heun's Method	Error
0.0	0.5000000	0.5000000	0
0.2	0.8292986	0.8292444	0.0000542
0.4	1.2140877	1.2139750	0.0001127
0.6	1.6489406	1.6487659	0.0001747
0.8	2.1272295	2.1269905	0.0002390
1.0	2.6408591	2.6405555	0.0003035
1.2	3.1799415	3.1795763	0.0003653
1.4	3.7324000	3.7319803	0.0004197
1.6	4.2834838	4.2830230	0.0004608
1.8	4.8151763	4.8146966	0.0004797
2.0	5.3054720	5.3050072	0.0004648

Higher-Order Runge-Kutta Methods

Runge-Kutta Order 4 Method

$$
\begin{aligned}
w_0 &= \alpha \\
k_1 &= hf(t_i, w_i) \\
k_2 &= hf\left(t_i + \frac{h}{2}, w_i + \frac{1}{2}k_1\right) \\
k_3 &= hf\left(t_i + \frac{h}{2}, w_i + \frac{1}{2}k_2\right) \\
k_4 &= hf(t_{i+1}, w_i + k_3) \\
w_{i+1} &= w_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)
\end{aligned}
$$

for each $i = 0, 1, \ldots, N-1$. This method has local truncation error $O(h^4)$, provided the solution $y(t)$ has five continuous derivatives.

Runge-Kutta Order 4 Algorithm (1/2)

To approximate the solution of the initial-value problem

$$y' = f(t, y), \quad a \leq t \leq b, \quad y(a) = \alpha$$

at $(N+1)$ equally spaced numbers in the interval $[a, b]$:

Runge-Kutta Order 4 Algorithm (1/2)

To approximate the solution of the initial-value problem

$$y' = f(t, y), \quad a \le t \le b, \quad y(a) = \alpha$$

at $(N + 1)$ equally spaced numbers in the interval $[a, b]$:

INPUT endpoints a, b; integer N; initial condition α.
OUTPUT approximation w to y at the $(N + 1)$ values of t.

Step 1 Set $h = (b - a)/N$
$t = a$
$w = \alpha$
OUTPUT (t, w)

Steps 2 to 6 on the next Slide

Runge-Kutta Order 4 Algorithm (2/2)

Step 2 For $i = 1, 2, \ldots, N$ do Steps 3–5:

Step 3 Set $K_1 = hf(t, w)$

$$K_2 = hf(t + h/2, w + K_1/2)$$
$$K_3 = hf(t + h/2, w + K_2/2)$$
$$K_4 = hf(t + h, w + K_3)$$

Step 4 Set $w = w + (K_1 + 2K_2 + 2K_3 + K_4)/6$

$$t = a + ih$$

Step 5 OUTPUT (t, w)

Step 6 STOP

Higher-Order Runge-Kutta Methods

Example: Runge-Kutta 4

Use the Runge-Kutta method of order four with $h = 0.2$, $N = 10$ and $t_i = 0.2i$ to obtain approximations to the solution of the initial-value problem

$$y' = y - t^2 + 1, \quad 0 \leq t \leq 2, \quad y(0) = 0.5$$

Higher-Order Runge-Kutta Methods

Solution

The approximation to $y(0.2)$ is obtained by

$$w_0 = 0.5$$

Higher-Order Runge-Kutta Methods

Solution

The approximation to $y(0.2)$ is obtained by

$$\begin{aligned} w_0 &= 0.5 \\ k_1 &= 0.2f(0, 0.5) = 0.2(1.5) = 0.3 \end{aligned}$$

Higher-Order Runge-Kutta Methods

Solution

The approximation to $y(0.2)$ is obtained by

$$\begin{aligned} w_0 &= 0.5 \\ k_1 &= 0.2f(0, 0.5) = 0.2(1.5) = 0.3 \\ k_2 &= 0.2f(0.1, 0.65) = 0.328 \end{aligned}$$

Higher-Order Runge-Kutta Methods

Solution

The approximation to $y(0.2)$ is obtained by

$$\begin{aligned}
w_0 &= 0.5 \\
k_1 &= 0.2f(0, 0.5) = 0.2(1.5) = 0.3 \\
k_2 &= 0.2f(0.1, 0.65) = 0.328 \\
k_3 &= 0.2f(0.1, 0.664) = 0.3308
\end{aligned}$$

Higher-Order Runge-Kutta Methods

Solution

The approximation to $y(0.2)$ is obtained by

$$\begin{aligned}
w_0 &= 0.5 \\
k_1 &= 0.2f(0, 0.5) = 0.2(1.5) = 0.3 \\
k_2 &= 0.2f(0.1, 0.65) = 0.328 \\
k_3 &= 0.2f(0.1, 0.664) = 0.3308 \\
k_4 &= 0.2f(0.2, 0.8308) = 0.35816
\end{aligned}$$

Higher-Order Runge-Kutta Methods

Solution

The approximation to $y(0.2)$ is obtained by

$$\begin{aligned} w_0 &= 0.5 \\ k_1 &= 0.2f(0, 0.5) = 0.2(1.5) = 0.3 \\ k_2 &= 0.2f(0.1, 0.65) = 0.328 \\ k_3 &= 0.2f(0.1, 0.664) = 0.3308 \\ k_4 &= 0.2f(0.2, 0.8308) = 0.35816 \\ w_1 &= 0.5 + \frac{1}{6}(0.3 + 2(0.328) + 2(0.3308) + 0.35816) = 0.8292933 \end{aligned}$$

Higher-Order Runge-Kutta Methods

Solution

The approximation to $y(0.2)$ is obtained by

$$\begin{aligned}
w_0 &= 0.5 \\
k_1 &= 0.2f(0, 0.5) = 0.2(1.5) = 0.3 \\
k_2 &= 0.2f(0.1, 0.65) = 0.328 \\
k_3 &= 0.2f(0.1, 0.664) = 0.3308 \\
k_4 &= 0.2f(0.2, 0.8308) = 0.35816 \\
w_1 &= 0.5 + \frac{1}{6}(0.3 + 2(0.328) + 2(0.3308) + 0.35816) = 0.8292933
\end{aligned}$$

The remaining results and their errors are listed in the following table.

Higher-Order Runge-Kutta Methods

t_i	Exact $y_i = y(t_i)$	Runge-Kutta Order Four w_i	Error $\lvert y_i - w_i \rvert$
0.0	0.5000000	0.5000000	0
0.2	0.8292986	0.8292933	0.0000053
0.4	1.2140877	1.2140762	0.0000114
0.6	1.6489406	1.6489220	0.0000186
0.8	2.1272295	2.1272027	0.0000269
1.0	2.6408591	2.6408227	0.0000364
1.2	3.1799415	3.1798942	0.0000474
1.4	3.7324000	3.7323401	0.0000599
1.6	4.2834838	4.2834095	0.0000743
1.8	4.8151763	4.8150857	0.0000906
2.0	5.3054720	5.3053630	0.0001089

A Comparison of Runge-Kutta Methods

Example

- For the problem

$$y' = y - t^2 + 1, \quad 0 \le t \le 2, \quad y(0) = 0.5$$

A Comparison of Runge-Kutta Methods

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$$y' = y - t^2 + 1, \quad 0 \le t \le 2, \quad y(0) = 0.5$$

Euler's method with $h = 0.025$, the Midpoint method with $h = 0.05$, and the Runge-Kutta 4th-order method with $h = 0.1$ are compared at the common mesh points of these methods 0.1, 0.2, 0.3, 0.4, and 0.5.

A Comparison of Runge-Kutta Methods

Example

- For the problem

$$y' = y - t^2 + 1, \quad 0 \le t \le 2, \quad y(0) = 0.5$$

  Euler's method with $h = 0.025$, the Midpoint method with $h = 0.05$, and the Runge-Kutta 4th-order method with $h = 0.1$ are compared at the common mesh points of these methods 0.1, 0.2, 0.3, 0.4, and 0.5.

- Each of these techniques requires 20 function evaluations to determine the values (listed in the following table) to approximate $y(0.5)$.

A Comparison of Runge-Kutta Methods

t_i	Exact	Euler $h = 0.025$	Modified Euler $h = 0.05$	Runge-Kutta Order Four $h = 0.1$
0.0	0.5000000	0.5000000	0.5000000	0.5000000
0.1	0.6574145	0.6554982	0.6573085	0.6574144
0.2	0.8292986	0.8253385	0.8290778	0.8292983
0.3	1.0150706	1.0089334	1.0147254	1.0150701
0.4	1.2140877	1.2056345	1.2136079	1.2140869
0.5	1.4256394	1.4147264	1.4250141	1.4256384

In this example, the fourth-order method is clearly superior.

Questions?