Initial-Value Problems for ODEs

Introduction to Linear Multistep Methods

Numerical Analysis (9th Edition)

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Outline

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From One-Step to Multistep Methods

The Nature of One-Step Methods

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- The methods met so far are called one-step methods because the approximation for the mesh point t_{i+1} involves information from only one of the previous mesh points, t_i.
- Although these methods might use function evaluation information at points between t_i and t_{i+1}, they do not retain that information for direct use in future approximations.
- All the information used by these methods is obtained within the subinterval over which the solution is being approximated.

From One-Step to Multistep Methods

Moving to Multistep Methods

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Moving to Multistep Methods

- The approximate solution is available at each of the mesh points $t_0, t_1, \ldots, t_i$ before the approximation at t_{i+1} is obtained, and because the error $|w_j - y(t_j)|$ tends to increase with j, . . .

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- so it seems reasonable to develop methods that use these more accurate previous data when approximating the solution at t_{i+1}.
- Methods using the approximation at more than one previous mesh point to determine the approximation at the next point are called multistep methods.

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- Methods using the approximation at more than one previous mesh point to determine the approximation at the next point are called multistep methods.
- We will now give a precise definition of these methods, together with the definition of the two types of multistep methods.

Outline

1. From One-Step to Multistep Methods

2. *m*-Step Multistep Methods

3. Example 4-Step Multistep Methods

An m-Step Multistep Method

An m-Step Multistep Method

Definition

An ***m*-step multistep method** for solving the initial-value problem

$$y' = f(t, y), \qquad a \le t \le b, \qquad y(a) = \alpha$$

has a difference equation for finding the approximation w_{i+1} at the mesh point t_{i+1} represented by the following equation, where m is an integer greater than 1:

$$\begin{aligned} w_{i+1} &= a_{m-1}w_i + a_{m-2}w_{i-1} + \cdots + a_0 w_{i+1-m} \\ &+ h\left[b_m f(t_{i+1}, w_{i+1}) + b_{m-1} f(t_i, w_i) + \cdots + b_0 f(t_{i+1-m}, w_{i+1-m})\right] \end{aligned}$$

for $i = m-1, m, \ldots, N-1$, where $h = (b-a)/N$, the $a_0, a_1, \ldots, a_{m-1}$ and $b_0, b_1, \ldots, b_m$ are constants, and the starting values

$$w_0 = \alpha, \quad w_1 = \alpha_1, \quad w_2 = \alpha_2, \quad \ldots, \quad w_{m-1} = \alpha_{m-1}$$

are specified.

An *m*-Step Multistep Method

$$\begin{aligned} w_{i+1} &= a_{m-1}w_i + a_{m-2}w_{i-1} + \cdots + a_0 w_{i+1-m} \\ &+ h\left[b_m f(t_{i+1}, w_{i+1}) + b_{m-1} f(t_i, w_i) + \cdots + b_0 f(t_{i+1-m}, w_{i+1-m})\right] \end{aligned}$$

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Explicit & Implicit Methods

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Explicit & Implicit Methods

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Explicit & Implicit Methods

- When $b_m = 0$, the method is called explicit, or open, because the difference equation then gives w_{i+1} explicitly in terms of previously determined values.
- When $b_m \neq 0$, the method is called implicit, or closed, because w_{i+1} occurs on both sides of the difference equation, so w_{i+1} is specified only implicitly.

Outline

The 4th-order Adams-Bashforth Method

The 4th-order Adams-Bashforth Method

The equations

$$
\begin{aligned}
w_0 &= \alpha \\
w_1 &= \alpha_1 \\
w_2 &= \alpha_2 \\
w_3 &= \alpha_3
\end{aligned}
$$

$$
\begin{aligned}
w_{i+1} &= w_i + \frac{h}{24}[55f(t_i, w_i) - 59f(t_{i-1}, w_{i-1}) + 37f(t_{i-2}, w_{i-2}) \\
&\qquad - 9f(t_{i-3}, w_{i-3})]
\end{aligned}
$$

for each $i = 3, 4, \ldots, N-1$, define an explicit 4-step method known as the 4th-order Adams-Bashforth technique.

The 4th-order Adams-Moulton Method

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The equations

$$
\begin{aligned}
w_0 &= \alpha \\
w_1 &= \alpha_1 \\
w_2 &= \alpha_2 \\
w_{i+1} &= w_i + \frac{h}{24}\left[9f(t_{i+1}, w_{i+1}) + 19f(t_i, w_i) - 5f(t_{i-1}, w_{i-1}) \right. \\
&\qquad \left. + f(t_{i-2}, w_{i-2})\right]
\end{aligned}
$$

for each $i = 2, 3, \ldots, N-1$, define an implicit 3-step method known as the 4th-order Adams-Moulton technique.

4th-Order Adams-Bashforth & Adams-Moulton Methods

Starting Values & Accuracy

4th-Order Adams-Bashforth & Adams-Moulton Methods

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- The 4 or 3 starting values (in either explicit AB4 or implicit AM4) must be specified, generally by assuming $w_0 = \alpha$ and generating the remaining values by either a Runge-Kutta or Taylor method.

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- The implicit methods are generally more accurate then the explicit methods, but to apply an implicit method such as the Adams-Moulton Method directly, we must solve the implicit equation for w_{i+1}.
- This is not always possible, and even when it can be done the solution for w_{i+1} may not be unique.

4th-order Adams-Bashforth Method

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- Assume that we have used the Runge-Kutta method of order 4 with $h = 0.2$ to approximate the solutions to the initial value problem

$$y' = y - t^2 + 1, \qquad 0 \leq t \leq 2, \qquad y(0) = 0.5$$

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- The first four approximations were found to be $y(0) = w_0 = 0.5$, $y(0.2) \approx w_1 = 0.8292933$, $y(0.4) \approx w_2 = 1.2140762$, and $y(0.6) \approx w_3 = 1.6489220$.

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- Use these as starting values for the 4th-order Adams-Bashforth method to compute new approximations for $y(0.8)$ and $y(1.0)$, and compare these new approximations to those produced by the Runge-Kutta method of order 4.

Example: 4th-order Adams-Bashforth Method

Solution (1/3)

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Solution (1/3)

For the 4th-order Adams-Bashforth, we have

$$\begin{aligned} y(0.8) \approx w_4 &= w_3 + \frac{0.2}{24}(55f(0.6, w_3) - 59f(0.4, w_2) + 37f(0.2, w_1) \\ &\quad -9f(0, w_0)) \end{aligned}$$

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Solution (2/3)

and

$$y(1.0) \approx w_5 = w_4 + \frac{0.2}{24}(55f(0.8, w_4) - 59f(0.6, w_3) + 37f(0.4, w_2) - 9f(0.2, w_1))$$

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&\qquad -9f(0.2, w_1)) \\
&= 2.1272892 + \frac{0.2}{24}(55f(0.8, 2.1272892) \\
&\qquad -59f(0.6, 1.6489220) + 37f(0.4, 1.2140762) \\
&\qquad -9f(0.2, 0.8292933)) \\
&= 2.1272892 + 0.0083333\,(55(2.4872892) \\
&\qquad -59(2.2889220) + 37(2.0540762) \\
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\end{aligned}$$

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&\qquad -9(1.7892933)) \\
&= 2.6410533
\end{aligned}
$$

Example: 4th-order Adams-Bashforth Method

Solution (3/3)

The error for these approximations at $t = 0.8$ and $t = 1.0$ are, respectively:

$$\begin{aligned} |2.1272295 - 2.1272892| &= 5.97 \times 10^{-5} \quad \text{and} \\ |2.6410533 - 2.6408591| &= 1.94 \times 10^{-4} \end{aligned}$$

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The corresponding Runge-Kutta approximations had errors:

$$\begin{aligned} |2.1272027 - 2.1272892| &= 2.69 \times 10^{-5} \quad \text{and} \\ |2.6408227 - 2.6408591| &= 3.64 \times 10^{-5} \end{aligned}$$

Questions?