Initial-Value Problems for ODEs

Variable Step-Size Multistep Methods

Numerical Analysis (9th Edition)

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Beamer Presentation Slides
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Outline

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Variable Step-Size Multistep Methods

Comparison with the RKF45 Method

Variable Step-Size Multistep Methods

Comparison with the RKF45 Method

- The Runge-Kutta-Fehlberg method is used for error control because at each step it provides, at little additional cost, two approximations that can be compared and related to the local truncation error.

Variable Step-Size Multistep Methods

Comparison with the RKF45 Method

- The Runge-Kutta-Fehlberg method is used for error control because at each step it provides, at little additional cost, two approximations that can be compared and related to the local truncation error.
- Predictor-corrector techniques always generate two approximations at each step, so they are natural candidates for error-control adaptation.

Variable Step-Size Multistep Methods

Comparison with the RKF45 Method

- The Runge-Kutta-Fehlberg method is used for error control because at each step it provides, at little additional cost, two approximations that can be compared and related to the local truncation error.
- Predictor-corrector techniques always generate two approximations at each step, so they are natural candidates for error-control adaptation.
- To demonstrate the error-control procedure, we construct a variable step-size predictor-corrector method using the 4-step explicit Adams-Bashforth method as predictor and the 3-step implicit Adams-Moulton method as corrector.

LTE of the Predictor-Corrector Method

4-step Explicit Adams-Bashforth Method

The Adams-Bashforth 4-step method comes from the relation

$$\begin{aligned} y(t_{i+1}) &= y(t_i) + \frac{h}{24}[55f(t_i, y(t_i)) - 59f(t_{i-1}, y(t_{i-1})) \\ &\quad + 37f(t_{i-2}, y(t_{i-2})) - 9f(t_{i-3}, y(t_{i-3}))] + \frac{251}{720}y^{(5)}(\hat{\mu}_i)h^5 \end{aligned}$$

for some $\hat{\mu}_i \in (t_{i-3}, t_{i+1})$.

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for some $\hat{\mu}_i \in (t_{i-3}, t_{i+1})$. The assumption that the approximations $w_0, w_1, \ldots, w_i$ are all exact implies that the Adams-Bashforth local truncation error (LTE) is

$$\frac{y(t_{i+1}) - w_{i+1}^p}{h} = \frac{251}{720}y^{(5)}(\hat{\mu}_i)h^4$$

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LTE of the Predictor-Corrector Method

3-Step Implicit Adams-Moulton Method

LTE of the Predictor-Corrector Method

3-Step Implicit Adams-Moulton Method

A similar analysis of the Adams-Moulton method which comes from

$$\begin{aligned} y(t_{i+1}) &= y(t_i) + \frac{h}{24}[9f(t_{i+1}, y(t_{i+1})) + 19f(t_i, y(t_i)) - 5f(t_{i-1}, y(t_{i-1})) \\ &\quad + f(t_{i-2}, y(t_{i-2}))] - \frac{19}{720}y^{(5)}(\tilde{\mu}_i)h^5 \end{aligned}$$

for some $\tilde{\mu}_i \in (t_{i-2}, t_{i+1})$,

LTE of the Predictor-Corrector Method

3-Step Implicit Adams-Moulton Method

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for some $\tilde{\mu}_i \in (t_{i-2}, t_{i+1})$, leads to the local truncation error (LTE):

$$\frac{y(t_{i+1}) - w^c_{i+1}}{h} = -\frac{19}{720} y^{(5)}(\tilde{\mu}_i) h^4$$

where w^c_{i+1} represents the corrected approximation given by the 3-Step implicit Adams-Moulton method.

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LTE of the Predictor-Corrector Method

Crucial Assumption

To proceed further, we must make the assumption that for small values of h, we have

$$y^{(5)}(\hat{\mu}_i) \approx y^{(5)}(\tilde{\mu}_i)$$

The effectiveness of the error-control technique depends directly on this assumption.

LTE of the Predictor-Corrector Method

Combining the LTE Estimates

If we subtract the AB4 (predictor) local truncation error estimate

$$\frac{y(t_{i+1}) - w_{i+1}^{p}}{h} = \frac{251}{720} y^{(5)}(\hat{\mu}_i) h^4$$

LTE of the Predictor-Corrector Method

Combining the LTE Estimates

If we subtract the AB4 (predictor) local truncation error estimate

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from the AM4 (corrector) local truncation error estimate

$$\frac{y(t_{i+1}) - w_{i+1}^{c}}{h} = -\frac{19}{720} y^{(5)}(\tilde{\mu}_i) h^4$$

LTE of the Predictor-Corrector Method

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from the AM4 (corrector) local truncation error estimate

$$\frac{y(t_{i+1}) - w_{i+1}^c}{h} = -\frac{19}{720} y^{(5)}(\tilde{\mu}_i) h^4$$

we have

$$\frac{w_{i+1}^c - w_{i+1}^p}{h} = \frac{h^4}{720}[251 y^{(5)}(\hat{\mu}_i) + 19 y^{(5)}(\tilde{\mu}_i)] \approx \frac{3}{8} h^4 y^{(5)}(\tilde{\mu}_i)$$

LTE of the Predictor-Corrector Method

$$\frac{w_{i+1}^c - w_{i+1}^p}{h} = \frac{h^4}{720}[251y^{(5)}(\hat{\mu}_i) + 19y^{(5)}(\tilde{\mu}_i)] \approx \frac{3}{8}h^4 y^{(5)}(\tilde{\mu}_i)$$

Combining the LTE Estimates (Cont'd)

Therefore

$$y^{(5)}(\tilde{\mu}_i) \approx \frac{8}{3h^5}(w_{i+1}^c - w_{i+1}^p)$$

LTE of the Predictor-Corrector Method

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Combining the LTE Estimates (Cont'd)

Therefore

$$y^{(5)}(\tilde{\mu}_i) \approx \frac{8}{3h^5}(w_{i+1}^c - w_{i+1}^p)$$

Using this result to eliminate the term involving $y^{(5)}(\tilde{\mu}_i)h^4$ in

$$\frac{y(t_{i+1}) - w_{i+1}^c}{h} = -\frac{19}{720}y^{(5)}(\tilde{\mu}_i)h^4$$

gives the following approximation to the Adams-Moulton local truncation error:

LTE of the Predictor-Corrector Method

Combining the LTE Estimates (Cont'd)

The approximation to the Adams-Moulton local truncation error is finally:

$$|\tau_{i+1}(h)| = \frac{|y(t_{i+1}) - w_{i+1}^c|}{h}$$

LTE of the Predictor-Corrector Method

Combining the LTE Estimates (Cont'd)

The approximation to the Adams-Moulton local truncation error is finally:

$$
\begin{aligned}
|\tau_{i+1}(h)| &= \frac{|y(t_{i+1}) - w^c_{i+1}|}{h} \\
&\approx \frac{19h^4}{720} \cdot \frac{8}{3h^5} |w^c_{i+1} - w^p_{i+1}|
\end{aligned}
$$

LTE of the Predictor-Corrector Method

Combining the LTE Estimates (Cont'd)

The approximation to the Adams-Moulton local truncation error is finally:

$$\begin{aligned} |\tau_{i+1}(h)| &= \frac{|y(t_{i+1}) - w_{i+1}^c|}{h} \\ &\approx \frac{19h^4}{720} \cdot \frac{8}{3h^5} |w_{i+1}^c - w_{i+1}^p| \\ &= \frac{19}{270h} \left|w_{i+1}^c - w_{i+1}^p\right| \end{aligned}$$

Outline

Variable Step-Size Multistep Methods

Changing the Step-Size

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Changing the Step-Size

- Suppose we now reconsider the 3-step AM4 Corrector LTE equation

$$\frac{y(t_{i+1}) - w_{i+1}^{c}}{h} = -\frac{19}{720} y^{(5)}(\tilde{\mu}_i) h^4$$

with a new step size qh generating new approximations $\hat{w}_{i+1}^{p}$ and $\hat{w}_{i+1}^{c}$.

Variable Step-Size Multistep Methods

Changing the Step-Size

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  with a new step size qh generating new approximations $\hat{w}_{i+1}^p$ and $\hat{w}_{i+1}^c$.
- The object is to choose q so that the local truncation error given in this equation is bounded by a prescribed tolerance ε.

Variable Step-Size Multistep Methods

Changing the Step-Size (Cont'd)

If we assume that the value $y^{(5)}(\mu)$ in the equation

$$\frac{y(t_{i+1}) - w_{i+1}^{c}}{h} = -\frac{19}{720} y^{(5)}(\tilde{\mu}_i) h^4$$

associated with qh

Variable Step-Size Multistep Methods

Changing the Step-Size (Cont'd)

If we assume that the value $y^{(5)}(\mu)$ in the equation

$$\frac{y(t_{i+1}) - w_{i+1}^{c}}{h} = -\frac{19}{720} y^{(5)}(\tilde{\mu}_i) h^4$$

associated with qh is also approximated using

$$y^{(5)}(\tilde{\mu}_i) \approx \frac{8}{3h^5}(w_{i+1}^{c} - w_{i+1}^{p})$$

Variable Step-Size Multistep Methods

Changing the Step-Size (Cont'd)

If we assume that the value $y^{(5)}(\mu)$ in the equation

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$$y^{(5)}(\tilde{\mu}_i) \approx \frac{8}{3h^5}(w_{i+1}^c - w_{i+1}^p)$$

then

$$\frac{|y(t_i + qh) - \hat{w}_{i+1}^c|}{qh} = \frac{19q^4h^4}{720}|y^{(5)}(\mu)| \approx \frac{19q^4h^4}{720}\left[\frac{8}{3h^5}|w_{i+1}^c - w_{i+1}^p|\right]$$

Variable Step-Size Multistep Methods

Changing the Step-Size (Cont'd)

If we assume that the value $y^{(5)}(\mu)$ in the equation

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associated with qh is also approximated using

$$y^{(5)}(\tilde{\mu}_i) \approx \frac{8}{3h^5}(w_{i+1}^c - w_{i+1}^p)$$

then

$$\begin{aligned} \frac{|y(t_i + qh) - \hat{w}_{i+1}^c|}{qh} = \frac{19q^4h^4}{720} |y^{(5)}(\mu)| &\approx \frac{19q^4h^4}{720} \left[\frac{8}{3h^5} |w_{i+1}^c - w_{i+1}^p| \right] \\ &= \frac{19q^4}{270} \frac{|w_{i+1}^c - w_{i+1}^p|}{h} \end{aligned}$$

Variable Step-Size Multistep Methods

Changing the Step-Size (Cont'd)

We need to choose q so that

$$\frac{|y(t_i + qh) - \hat{w}^c_{i+1}|}{qh} \approx \frac{19q^4}{270} \frac{|w^c_{i+1} - w^p_{i+1}|}{h} < \varepsilon$$

Variable Step-Size Multistep Methods

Changing the Step-Size (Cont'd)

We need to choose q so that

$$\frac{|y(t_i + qh) - \hat{w}_{i+1}^c|}{qh} \approx \frac{19q^4}{270} \frac{|w_{i+1}^c - w_{i+1}^p|}{h} < \varepsilon$$

That is, choose q so that

$$q < \left(\frac{270}{19} \frac{h\varepsilon}{|w_{i+1}^c - w_{i+1}^p|} \right)^{1/4} \approx 2 \left(\frac{h\varepsilon}{|w_{i+1}^c - w_{i+1}^p|} \right)^{1/4}$$

Variable Step-Size Multistep Methods

$$q < \left(\frac{270}{19} \frac{h\varepsilon}{|w_{i+1}^{c} - w_{i+1}^{p}|} \right)^{1/4} \approx 2 \left(\frac{h\varepsilon}{|w_{i+1}^{c} - w_{i+1}^{p}|} \right)^{1/4}$$

Variable Step-Size Multistep Methods

$$q < \left(\frac{270}{19} \frac{h\varepsilon}{|w_{i+1}^c - w_{i+1}^p|} \right)^{1/4} \approx 2 \left(\frac{h\varepsilon}{|w_{i+1}^c - w_{i+1}^p|} \right)^{1/4}$$

Changing the Step-Size (Cont'd)

- A number of approximation assumptions have been made in this development, so in practice q is chosen conservatively, often as

$$q = 1.5 \left(\frac{h\varepsilon}{|w_{i+1}^c - w_{i+1}^p|} \right)^{1/4}$$

Variable Step-Size Multistep Methods

The Cost of a Change in Step-Size

- A change in step size for a multistep method is more costly in terms of function evaluations than for a one-step method, because new equally-spaced starting values must be computed.

Variable Step-Size Multistep Methods

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- A change in step size for a multistep method is more costly in terms of function evaluations than for a one-step method, because new equally-spaced starting values must be computed.
- As a consequence, it is common practice to ignore the step-size change whenever the local truncation error is between $\varepsilon/10$ and ε, that is, when

$$\frac{\varepsilon}{10} < |\tau_{i+1}(h)| = \frac{|y(t_{i+1}) - w_{i+1}^c|}{h} \approx \frac{19\left|w_{i+1}^c - w_{i+1}^p\right|}{270h} < \varepsilon$$

Variable Step-Size Multistep Methods

The Cost of a Change in Step-Size

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- In addition, q is given an upper bound (e.g. 4) to ensure that a single unusually accurate approximation does not result in too large a step size.

Outline

Adams Variable Step-Size P-C Algorithm (1/7)

To approximate the solution of the initial-value problem

$$y' = f(t, y), \qquad a \le t \le b, \qquad y(a) = \alpha$$

with local truncation error within a given tolerance:

Adams Variable Step-Size P-C Algorithm (1/7)

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$$y' = f(t, y), \qquad a \le t \le b, \qquad y(a) = \alpha$$

with local truncation error within a given tolerance:

INPUT endpoints a, b; initial condition α; tolerance TOL; maximum step size $hmax$; minimum step size $hmin$

OUTPUT i, t_i, w_i, h where at the ith step w_i approximates $y(t_i)$ and the step size h was used, or a message that the minimum step size was exceeded.

Adams Variable Step-Size P-C Algorithm (2/7)

Step 1 Set up a subalgorithm for the Runge-Kutta 4th-order method to be called $RK4(h, v_0, x_0, v_1, x_1, v_2, x_2, v_3, x_3)$ that accepts as input a step size h and starting values $v_0 \approx y(x_0)$ and returns $\{(x_j, v_j) \mid j = 1, 2, 3\}$ defined by the following:

for $j = 1, 2, 3$

set $K_1 = hf(x_{j-1}, v_{j-1})$

$K_2 = hf(x_{j-1} + h/2, v_{j-1} + K_1/2)$

$K_3 = hf(x_{j-1} + h/2, v_{j-1} + K_2/2)$

$K_4 = hf(x_{j-1} + h, v_{j-1} + K_3)$

$v_j = v_{j-1} + (K_1 + 2K_2 + 2K_3 + K_4)/6$

$x_j = x_0 + jh$

Adams Variable Step-Size P-C Algorithm (3/7)

Step 2 Set $t_0 = a$

$w_0 = \alpha$

$h = hmax$

$FLAG = 1$

(*FLAG will be used to exit the loop in Step 4*)

$LAST = 0$

(*LAST will indicate when the last value is calculated*)

OUTPUT (t_0, w_0)

Adams Variable Step-Size P-C Algorithm (3/7)

Step 2 Set $t_0 = a$

$w_0 = \alpha$

$h = hmax$

$FLAG = 1$

(*FLAG will be used to exit the loop in Step 4*)

$LAST = 0$

(*LAST will indicate when the last value is calculated*)

OUTPUT (t_0, w_0)

Step 3 Call $RK4(h, w_0, t_0, w_1, t_1, w_2, t_2, w_3, t_3)$

Set $NFLAG = 1$ (*Indicates computation from RK4*)

$i = 4$

$t = t_3 + h$

Adams Variable Step-Size P-C Algorithm (4/7)

Step 4 While ($FLAG = 1$) do Steps 5–20

Adams Variable Step-Size P-C Algorithm (4/7)

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Step 5 Set
$$WP = w_{i-1} + \frac{h}{24}[55f(t_{i-1}, w_{i-1}) - 59f(t_{i-2}, w_{i-2}) + 37f(t_{i-3}, w_{i-3}) - 9f(t_{i-4}, w_{i-4})] \text{ (Predict } w_i)$$

$$WC = w_{i-1} + \frac{h}{24}[9f(t, WP) + 19f(t_{i-1}, w_{i-1}) - 5f(t_{i-2}, w_{i-2}) + f(t_{i-3}, w_{i-3})] \text{ (Correct } w_i)$$

$$\sigma = 19|WC - WP|/(270h)$$

Adams Variable Step-Size P-C Algorithm (4/7)

Step 4 While ($FLAG = 1$) do Steps 5–20

Step 5 Set $WP = w_{i-1} + \frac{h}{24}[55f(t_{i-1}, w_{i-1}) - 59f(t_{i-2}, w_{i-2}) + 37f(t_{i-3}, w_{i-3}) - 9f(t_{i-4}, w_{i-4})]$ (*Predict* w_i)

$$WC = w_{i-1} + \frac{h}{24}[9f(t, WP) + 19f(t_{i-1}, w_{i-1}) - 5f(t_{i-2}, w_{i-2}) + f(t_{i-3}, w_{i-3})] \quad (Correct \;\; w_i)$$

$$\sigma = 19|WC - WP|/(270h)$$

Step 6 If $\sigma \leq TOL$ then do Steps 7–16 (*Result accepted*)
else do Steps 17–19 (*Result rejected*)

Adams Variable Step-Size P-C Algorithm (5/7)

Step 6 (Continuation of Steps 7–16)

Step 7 Set $w_i = WC$ (*Result accepted*)

$t_i = t$

Adams Variable Step-Size P-C Algorithm (5/7)

Step 6 (Continuation of Steps 7–16)

Step 7 Set $w_i = WC$ (*Result accepted*)

$t_i = t$

Step 8 If $NFLAG = 1$ then for $j = i - 3, i - 2, i - 1, i$

OUTPUT (j, t_j, w_j, h)

(*Previous results also accepted*)

else OUTPUT (i, t_i, w_i, h)

(*Previous results already accepted*)

Adams Variable Step-Size P-C Algorithm (5/7)

Step 6 (Continuation of Steps 7–16)

Step 7 Set $w_i = WC$ (*Result accepted*)
$t_i = t$

Step 8 If $NFLAG = 1$ then for $j = i - 3, i - 2, i - 1, i$
OUTPUT (j, t_j, w_j, h)
(*Previous results also accepted*)
else OUTPUT (i, t_i, w_i, h)
(*Previous results already accepted*)

Step 9 If $LAST = 1$ then set $FLAG = 0$ (*Next step is* 20)
else do Steps 10–16

Adams Variable Step-Size P-C Algorithm (5/7)

Step 6 (Continuation of Steps 7–16)

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$t_i = t$

Step 8 If $NFLAG = 1$ then for $j = i-3, i-2, i-1, i$
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else OUTPUT (i, t_i, w_i, h)
(*Previous results already accepted*)

Step 9 If $LAST = 1$ then set $FLAG = 0$ (*Next step is* 20)
else do Steps 10–16

Step 10 Set $i = i + 1$
$NFLAG = 0$

Adams Variable Step-Size P-C Algorithm (6/7)

Step 6 (Continuation of Steps 7–16)

Step 11 If $\sigma \leq 0.1\,TOL$ or $t_{i-1} + h > b$ then do Steps 12–16 (*Increase h if it is more accurate than required or decrease h to include b as a mesh point*)

Adams Variable Step-Size P-C Algorithm (6/7)

Step 6 (Continuation of Steps 7–16)

Step 11 If $\sigma \leq 0.1\ TOL$ or $t_{i-1} + h > b$ then do Steps 12–16
(*Increase h if it is more accurate than required or decrease h to include b as a mesh point*)

Step 12 Set $q = (TOL/(2\sigma))^{1/4}$

Adams Variable Step-Size P-C Algorithm (6/7)

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else set $h = qh$

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Step 13 If $q > 4$ then set $h = 4h$
else set $h = qh$

Step 14 If $h > hmax$ then set $h = hmax$

Adams Variable Step-Size P-C Algorithm (6/7)

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else set $h = qh$

Step 14 If $h > hmax$ then set $h = hmax$

Step 15 If $t_{i-1} + 4h > b$ then
set $h = (b - t_{i-1})/4$ & $LAST = 1$

Adams Variable Step-Size P-C Algorithm (6/7)

Step 6 (Continuation of Steps 7–16)

Step 11 If $\sigma \leq 0.1\ TOL$ or $t_{i-1} + h > b$ then do Steps 12–16
(*Increase h if it is more accurate than required or decrease h to include b as a mesh point*)

Step 12 Set $q = (TOL/(2\sigma))^{1/4}$

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else set $h = qh$

Step 14 If $h > hmax$ then set $h = hmax$

Step 15 If $t_{i-1} + 4h > b$ then
set $h = (b - t_{i-1})/4$ & $LAST = 1$

Step 16 Call $RK4(h, w_{i-1}, t_{i-1}, w_i, t_i, w_{i+1}, t_{i+1}, w_{i+2}, t_{i+2})$
Set $NFLAG = 1$
$i = i + 3$
(*True branch completed. Next step is 20*)

Adams Variable Step-Size P-C Algorithm (7/7)

Steps 17–19 (*False branch from Step 6: Result rejected*)

Step 17 Set $q = (TOL/(2\sigma))^{1/4}$

Adams Variable Step-Size P-C Algorithm (7/7)

Steps 17–19 (*False branch from Step 6: Result rejected*)

Step 17 Set $q = (TOL/(2\sigma))^{1/4}$

Step 18 If $q < 0.1$ then set $h = 0.1h$

else set $h = qh$

Adams Variable Step-Size P-C Algorithm (7/7)

Steps 17–19 (*False branch from Step 6: Result rejected*)

Step 17 Set $q = (TOL/(2\sigma))^{1/4}$

Step 18 If $q < 0.1$ then set $h = 0.1h$
else set $h = qh$

Step 19 If $h < hmin$ then set $FLAG = 0$
OUTPUT (‘*hmin* exceeded’)
else if $NFLAG = 1$ then set $i = i - 3$
(*Previous results also rejected*)
Call $RK4(h, w_{i-1}, t_{i-1}, w_i, t_i, w_{i+1}, t_{i+1}, w_{i+2}, t_{i+2})$
set $i = i + 3$
$NFLAG = 1$

Adams Variable Step-Size P-C Algorithm (7/7)

Steps 17–19 (*False branch from Step 6: Result rejected*)

Step 17 Set $q = (TOL/(2\sigma))^{1/4}$

Step 18 If $q < 0.1$ then set $h = 0.1h$
else set $h = qh$

Step 19 If $h < hmin$ then set $FLAG = 0$
OUTPUT (‘*hmin* exceeded’)
else if $NFLAG = 1$ then set $i = i - 3$
(*Previous results also rejected*)
Call $RK4(h, w_{i-1}, t_{i-1}, w_i, t_i, w_{i+1}, t_{i+1}, w_{i+2}, t_{i+2})$
set $i = i + 3$
$NFLAG = 1$

Step 20 Set $t = t_{i-1} + h$

Step 21 STOP

Adams Variable Step-Size P-C Algorithm: Example

Example: Applying the Algorithm

Use the Adams variable step-size predictor-corrector method with maximum step size $hmax = 0.2$, minimum step size $hmin = 0.01$, and tolerance $TOL = 10^{-5}$ to approximate the solution of the initial-value problem:

$$y' = y - t^2 + 1, \qquad 0 \le t \le 2, \qquad y(0) = 0.5$$

Adams Variable Step-Size P-C Algorithm: Example

Solution (1/5)

Adams Variable Step-Size P-C Algorithm: Example

Solution (1/5)

- We begin with $h = hmax = 0.2$, and obtain w_0, w_1, w_2 and w_3 using Runge-Kutta, then find w_4^p and w_4^c by applying the predictor-corrector method.

Adams Variable Step-Size P-C Algorithm: Example

Solution (1/5)

- We begin with $h = hmax = 0.2$, and obtain w_0, w_1, w_2 and w_3 using Runge-Kutta, then find w_4^p and w_4^c by applying the predictor-corrector method.
- These calculations were done in an earlier example where it was determined that the Runge-Kutta approximations are:

$$\begin{aligned} y(0) = w_0 &= 0.5 \\ y(0.2) \approx w_1 &= 0.8292933 \\ y(0.4) \approx w_2 &= 1.2140762 \quad \text{and} \\ y(0.6) \approx w_3 &= 1.6489220 \end{aligned}$$

Adams Variable Step-Size P-C Algorithm: Example

Solution (2/5)

The predictor and corrector gave

$$
\begin{aligned}
y(0) = w_0 &= 0.5 \\
y(0.2) \approx w_1 &= 0.8292933 \\
y(0.4) \approx w_2 &= 1.2140762 \\
y(0.6) \approx w_3 &= 1.6489220 \\
y(0.8) \approx w_4^p &= w_3 + \frac{0.2}{24}\left(55f(0.6, w_3) - 59f(0.4, w_2) + 37f(0.2, w_1)\right. \\
&\qquad \left. -9f(0, w_0)\right) \\
&= 2.1272892 \\
y(0.8) \approx w_4^c &= w_3 + \frac{0.2}{24}\left(9f(0.8, w_4^p) + 19f(0.6, w_3) - 5f(0.42, w_2)\right. \\
&\qquad \left. +f(0.2, w_1)\right) \\
&= 2.1272056
\end{aligned}
$$

Adams Variable Step-Size P-C Algorithm: Example

Solution (3/5)

We now need to determine if these approximations are sufficiently accurate or if there needs to be a change in the step size.

Adams Variable Step-Size P-C Algorithm: Example

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$$\delta = \frac{19}{270h}|w_4^c - w_4^p| = \frac{19}{270(0.2)}|2.1272056 - 2.1272892| = 2.941 \times 10^{-5}.$$

Adams Variable Step-Size P-C Algorithm: Example

Solution (3/5)

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$$\delta = \frac{19}{270h}|w_4^c - w_4^p| = \frac{19}{270(0.2)}|2.1272056 - 2.1272892| = 2.941 \times 10^{-5}.$$

Because this exceeds the tolerance of 10^{-5}, a new step size is needed and the new step size is:

$$qh = \left(\frac{10^{-5}}{2\delta}\right)^{1/4} = \left(\frac{10^{-5}}{2(2.941 \times 10^{-5})}\right)^{1/4}(0.2) = 0.642(0.2) \approx 0.128.$$

Adams Variable Step-Size P-C Algorithm: Example

Solution (4/5)

Adams Variable Step-Size P-C Algorithm: Example

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Adams Variable Step-Size P-C Algorithm: Example

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Adams Variable Step-Size P-C Algorithm: Example

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- We then need to run the accuracy check on these approximations to see that we have been successful.

Adams Variable Step-Size P-C Algorithm: Example

Solution (4/5)

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- and then use the predictor-corrector method with this same step size to compute the new values of w_4^p and w_4^c.
- We then need to run the accuracy check on these approximations to see that we have been successful.
- The following table shows that this second run is successful and lists the all results obtained using the algorithm.

Adams Variable Step-Size P-C Algorithm: Example

Solution (5/5): Summary of Numerical Results

t_i	$y(t_i)$	w_i	h_i	σ_i	$\lvert y(t_i) - w_i \rvert$
0	0.5	0.5			
0.1257017	0.7002323	0.7002318	0.1257017	4.051×10^{-6}	0.0000005
0.2514033	0.9230960	0.9230949	0.1257017	4.051×10^{-6}	0.0000011
0.3771050	1.1673894	1.1673877	0.1257017	4.051×10^{-6}	0.0000017
⋮	⋮	⋮	⋮	⋮	⋮
1.2570166	3.3366642	3.3366562	0.1257017	8.622×10^{-6}	0.0000080
1.3827183	3.6844857	3.6844761	0.1257017	9.777×10^{-6}	0.0000097
1.4857283	3.9697541	3.9697433	0.1030100	7.029×10^{-6}	0.0000108
1.5887383	4.2527830	4.2527711	0.1030100	7.029×10^{-6}	0.0000120
⋮	⋮	⋮	⋮	⋮	⋮
1.8977683	5.0615660	5.0615488	0.1030100	7.760×10^{-6}	0.0000172
1.9233262	5.1239941	5.1239764	0.0255579	3.918×10^{-8}	0.0000177
1.9488841	5.1854932	5.1854751	0.0255579	3.918×10^{-8}	0.0000181
1.9744421	5.2460056	5.2459870	0.0255579	3.918×10^{-8}	0.0000186
2.0000000	5.3054720	5.3054529	0.0255579	3.918×10^{-8}	0.0000191

Questions?