Iterative Techniques in Matrix Algebra

Norms of Vectors & Matrices

Numerical Analysis (9th Edition)

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Beamer Presentation Slides
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Outline

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Norms of Vectors & Matrices: Introduction

The Need for Building Blocks

- To discuss iterative methods for solving linear systems, we first need to determine a way to measure the distance between n-dimensional column vectors.

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- This measure is also needed when the solution is obtained by the direct methods presented earlier.

Norms of Vectors & Matrices: Introduction

The Need for Building Blocks

- To discuss iterative methods for solving linear systems, we first need to determine a way to measure the distance between n-dimensional column vectors.
- This will permit us to determine whether a sequence of vectors converges to a solution of the system.
- This measure is also needed when the solution is obtained by the direct methods presented earlier.
- Those methods required a large number of arithmetic operations, and using finite-digit arithmetic leads only to an approximation to an actual solution of the system.

Norms of Vectors & Matrices: Vector Norms

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(iv) $\|\mathbf{x}+\mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.

Norms of Vectors & Matrices: Vector Norms

Comment

- Vectors in $\mathbb{R}^n$ are column vectors, and it is convenient to use the transpose notation (presented earlier) when a vector is represented in terms of its components.

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- For example, the vector

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

will be written $\mathbf{x} = (x_1, x_2, \ldots, x_n)^t$.

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The l_2 and l_∞ norms for the vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)^t$ are defined by

$$\|\mathbf{x}\|_2 = \left\{ \sum_{i=1}^{n} x_i^2 \right\}^{1/2} \quad \text{and} \quad \|\mathbf{x}\|_\infty = \max_{1 \le i \le n} |x_i|$$

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Note that each of these norms reduces to the absolute value in the case $n = 1$.

Norms of Vectors & Matrices: Vector Norms

Comments on the l_2 and l_∞ Norms

- The l_2 norm is called the Euclidean norm of the vector $\mathbf{x}$ because it represents the usual notion of distance from the origin in case $\mathbf{x}$ is in $\mathbb{R}^1 \equiv \mathbb{R}$, $\mathbb{R}^2$, or $\mathbb{R}^3$.

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- For example, the l_2 norm of the vector $\mathbf{x} = (x_1, x_2, x_3)^t$ gives the length of the straight line joining the points $(0, 0, 0)$ and (x_1, x_2, x_3).

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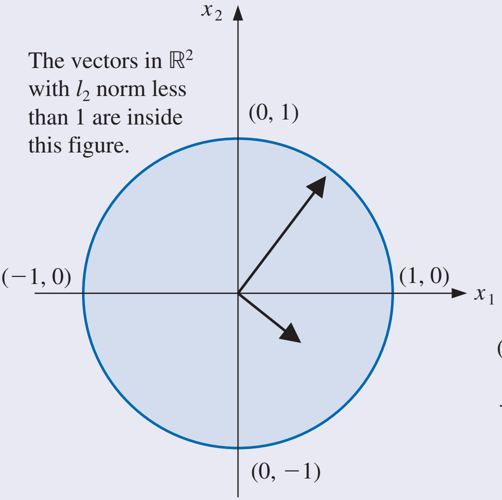
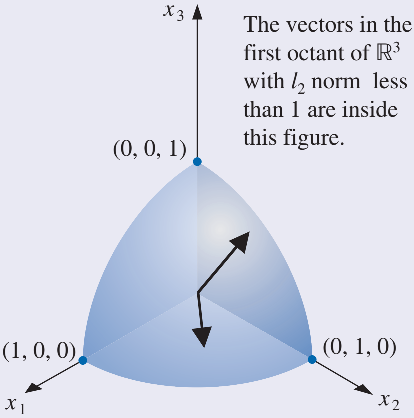
- For example, the l_2 norm of the vector $\mathbf{x} = (x_1, x_2, x_3)^t$ gives the length of the straight line joining the points $(0, 0, 0)$ and (x_1, x_2, x_3).
- The following 2 diagrams show the boundary of those vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$ that have l_2 and l_∞ norms respectively less than 1.

Boundary in $\mathbb{R}^2$ and $\mathbb{R}^3$ with $l_2 < 1$

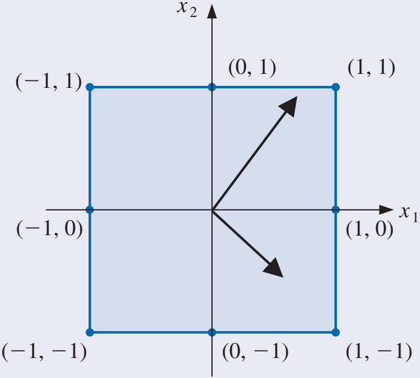
The vectors in $\mathbb{R}^2$ with l_2 norm less than 1 are inside this figure.



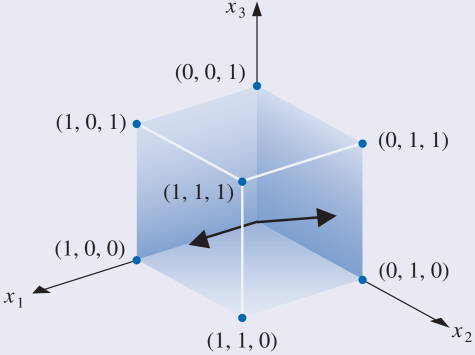
The vectors in the first octant of $\mathbb{R}^3$ with l_2 norm less than 1 are inside this figure.



Boundary in $\mathbb{R}^2$ and $\mathbb{R}^3$ with $l_\infty < 1$



The vectors in $\mathbb{R}^2$ with l_∞ norm less than 1 are inside this figure.



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Example

Determine the l_2 norm and the l_∞ norm of the vector $\mathbf{x} = (-1, 1, -2)^t$.

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The vector $\mathbf{x} = (-1, 1, -2)^t$ in $\mathbb{R}^3$ has norms

$$\|\mathbf{x}\|_2 = \sqrt{(-1)^2 + (1)^2 + (-2)^2} = \sqrt{6}$$

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and

$$\|\mathbf{x}\|_\infty = \max\{|-1|, |1|, |-2|\} = 2$$

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Establishing the Properties of a Vector Norm

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- The only property that requires much demonstration is (iv), i.e.

$$\|\mathbf{x}+\mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\| \qquad \forall \ \mathbf{x}, \mathbf{y} \in \mathbb{R}^n$$

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$$\|\mathbf{x}+\mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\| \qquad \forall \;\; \mathbf{x}, \mathbf{y} \in \mathbb{R}^n$$

- In this case, if $\mathbf{x} = (x_1, x_2, \ldots, x_n)^t$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)^t$, then

$$\|\mathbf{x}+\mathbf{y}\|_\infty$$

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$$\begin{aligned} \|\mathbf{x}+\mathbf{y}\|_\infty &= \max_{1\leq i\leq n} |x_i + y_i| \leq \max_{1\leq i\leq n} (|x_i| + |y_i|) \\ &\leq \max_{1\leq i\leq n} |x_i| + \max_{1\leq i\leq n} |y_i| \end{aligned}$$

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Norms of Vectors & Matrices: Vector Norms

Establishing the Properties of a Vector Norm (Cont'd)

- The first three conditions also are easy to show for the l_2 norm.

Norms of Vectors & Matrices: Vector Norms

Establishing the Properties of a Vector Norm (Cont'd)

- The first three conditions also are easy to show for the l_2 norm.
- But to show that

$$\|\mathbf{x}+\mathbf{y}\|_2 \leq \|\mathbf{x}\|_2 + \|\mathbf{y}\|_2, \quad \text{for each } \mathbf{x}, \mathbf{y} \in \mathbb{R}_n$$

we need a famous inequality.

Norms of Vectors & Matrices: Vector Norms

Theorem (Cauchy-Bunyakovsky-Schwarz Inequality for Sums)

For each $\mathbf{x} = (x_1, x_2, \ldots, x_n)^t$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)^t$ in $\mathbb{R}^n$,

$$\mathbf{x}^t\mathbf{y} = \sum_{i=1}^{n} x_i y_i \leq \left\{\sum_{i=1}^{n} x_i^2\right\}^{1/2} \left\{\sum_{i=1}^{n} y_i^2\right\}^{1/2} = \|\mathbf{x}\|_2 \cdot \|\mathbf{y}\|_2$$

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Note: If $\mathbf{y} = \mathbf{0}$ or $\mathbf{x} = \mathbf{0}$, the result is immediate because both sides of the inequality are zero.

Norms of Vectors & Matrices: Vector Norms

Proof (1/2)

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$$0 \leq ||\mathbf{x} - \lambda\mathbf{y}||_2^2 = \sum_{i=1}^{n}(x_i - \lambda y_i)^2 = \sum_{i=1}^{n} x_i^2 - 2\lambda \sum_{i=1}^{n} x_i y_i + \lambda^2 \sum_{i=1}^{n} y_i^2$$

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so that

$$2\lambda \sum_{i=1}^{n} x_i y_i \leq \sum_{i=1}^{n} x_i^2 + \lambda^2 \sum_{i=1}^{n} y_i^2 = \|\mathbf{x}\|_2^2 + \lambda^2 \|\mathbf{y}\|_2^2$$

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so that

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However $\|\mathbf{x}\|_2 > 0$ and $\|\mathbf{y}\|_2 > 0$, so we can let $\lambda = \|\mathbf{x}\|_2 / \|\mathbf{y}\|_2$ to give

$$\left(2 \frac{\|\mathbf{x}\|_2}{\|\mathbf{y}\|_2}\right) \left(\sum_{i=1}^{n} x_i y_i\right) \leq \|\mathbf{x}\|_2^2 + \frac{\|\mathbf{x}\|_2^2}{\|\mathbf{y}\|_2^2} \|\mathbf{y}\|_2^2 = 2\|\mathbf{x}\|_2^2$$

Norms of Vectors & Matrices: Vector Norms

Proof (2/2)

Hence

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Norms of Vectors & Matrices: Vector Norms

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and

$$\mathbf{x}^t\mathbf{y} = \sum_{i=1}^{n} x_i y_i \le \|\mathbf{x}\|_2\|\mathbf{y}\|_2 = \left\{\sum_{i=1}^{n} x_i^2\right\}^{1/2} \left\{\sum_{i=1}^{n} y_i^2\right\}^{1/2}$$

Norms of Vectors & Matrices: Vector Norms

Using the Cauchy-Bunyakovsky-Schwarz Inequality for Sums

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With this result we see that for each $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$,

$$\|\mathbf{x} + \mathbf{y}\|_2^2 = \sum_{i=1}^{n} (x_i + y_i)^2$$

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$$\begin{aligned}\|\mathbf{x}+\mathbf{y}\|_2^2 &= \sum_{i=1}^{n}(x_i+y_i)^2 \\ &= \sum_{i=1}^{n} x_i^2 + 2\sum_{i=1}^{n} x_i y_i + \sum_{i=1}^{n} y_i^2\end{aligned}$$

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&\leq \|\mathbf{x}\|_2^2 + 2\|\mathbf{x}\|_2\|\mathbf{y}\|_2 + \|\mathbf{y}\|_2^2
\end{aligned}$$

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&= \sum_{i=1}^{n} x_i^2 + 2\sum_{i=1}^{n} x_i y_i + \sum_{i=1}^{n} y_i^2 \\
&\leq \|\mathbf{x}\|_2^2 + 2\|\mathbf{x}\|_2\|\mathbf{y}\|_2 + \|\mathbf{y}\|_2^2
\end{aligned}$$

which gives norm property (iv):

$$\|\mathbf{x}+\mathbf{y}\|_2 \leq \left(\|\mathbf{x}\|_2^2 + 2\|\mathbf{x}\|_2\|\mathbf{y}\|_2 + \|\mathbf{y}\|_2^2\right)^{1/2} = \|\mathbf{x}\|_2 + \|\mathbf{y}\|_2$$

Outline

Norms of Vectors: Distance between Vectors in $\mathbb{R}^n$

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Definition: Distance between Vectors

If $\mathbf{x} = (x_1, x_2, \ldots, x_n)^t$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)^t$ are vectors in $\mathbb{R}^n$, the l_2 and l_∞ distances between $\mathbf{x}$ and $\mathbf{y}$ are defined by

$$\|\mathbf{x} - \mathbf{y}\|_2 = \left\{ \sum_{i=1}^{n} (x_i - y_i)^2 \right\}^{1/2} \quad \text{and} \quad \|\mathbf{x} - \mathbf{y}\|_\infty = \max_{1 \le i \le n} |x_i - y_i|$$

Norms of Vectors: Distance between Vectors in $\mathbb{R}^n$

Example

The linear system

$$\begin{aligned} 3.3330x_1 + 15920x_2 - 10.333x_3 &= 15913 \\ 2.2220x_1 + 16.710x_2 + 9.6120x_3 &= 28.544 \\ 1.5611x_1 + 5.1791x_2 + 1.6852x_3 &= 8.4254 \end{aligned}$$

has the exact solution $\mathbf{x} = (x_1, x_2, x_3)^t = (1, 1, 1)^t$,

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\end{aligned}
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has the exact solution $\mathbf{x} = (x_1, x_2, x_3)^t = (1, 1, 1)^t$, and Gaussian elimination, performed using five-digit rounding arithmetic and partial pivoting, produces the approximate solution

$$\tilde{\mathbf{x}} = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)^t = (1.2001, 0.99991, 0.92538)^t$$

Norms of Vectors: Distance between Vectors in $\mathbb{R}^n$

Example

The linear system

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Determine the l_2 and l_∞ distances between the exact and approximate solutions.

Norms of Vectors: Distance between Vectors in $\mathbb{R}^n$

Solution

Measurements of $\mathbf{x} - \tilde{\mathbf{x}}$ are given by

$$\begin{aligned} \|\mathbf{x} - \tilde{\mathbf{x}}\|_\infty &= \max\{|1 - 1.2001|, |1 - 0.99991|, |1 - 0.92538|\} \\ &= \max\{0.2001, 0.00009, 0.07462\} = 0.2001 \end{aligned}$$

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and

$$\begin{aligned} \|\mathbf{x} - \tilde{\mathbf{x}}\|_2 &= \left[(1 - 1.2001)^2 + (1 - 0.99991)^2 + (1 - 0.92538)^2\right]^{1/2} \\ &= [(0.2001)^2 + (0.00009)^2 + (0.07462)^2]^{1/2} = 0.21356 \end{aligned}$$

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Although the components $\tilde{x}_2$ and $\tilde{x}_3$ are good approximations to x_2 and x_3, the component $\tilde{x}_1$ is a poor approximation to x_1, and $|x_1 - \tilde{x}_1|$ dominates both norms.

Norms of Vectors: Distance between Vectors in $\mathbb{R}^n$

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Definition

A sequence $\{\mathbf{x}^{(k)}\}_{k=1}^{\infty}$ of vectors in $\mathbb{R}^n$ is said to converge to $\mathbf{x}$ with respect to the norm $\|\cdot\|$ if, given any $\varepsilon > 0$, there exists an integer $N(\varepsilon)$ such that

$$\|\mathbf{x}^{(k)} - \mathbf{x}\| < \varepsilon, \quad \text{for all } k \geq N(\varepsilon)$$

Norms of Vectors: Distance between Vectors in $\mathbb{R}^n$

Theorem

The sequence of vectors $\{\mathbf{x}^{(k)}\}$ converges to $\mathbf{x}$ in $\mathbb{R}^n$ with respect to the l_∞ norm if and only if

$$\lim_{k\to\infty} x_i^{(k)} = x_i$$

for each $i = 1, 2, \ldots, n$.

Norms of Vectors: Distance between Vectors in $\mathbb{R}^n$

Proof (1/2)

Suppose $\{\mathbf{x}^{(k)}\}$ converges to $\mathbf{x}$ with respect to the l_∞ norm.

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Suppose $\{\mathbf{x}^{(k)}\}$ converges to $\mathbf{x}$ with respect to the l_∞ norm. Given any $\varepsilon > 0$, there exists an integer $N(\varepsilon)$ such that for all $k \geq N(\varepsilon)$,

$$\max_{i=1,2,\ldots,n} |x_i^{(k)} - x_i| = \|\mathbf{x}^{(k)} - \mathbf{x}\|_\infty < \varepsilon$$

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This result implies that $|x_i^{(k)} - x_i| < \varepsilon$, for each $i = 1, 2, \ldots, n$, so $\lim_{k\to\infty} x_i^{(k)} = x_i$ for each i.

Norms of Vectors: Distance between Vectors in $\mathbb{R}^n$

Proof (2/2)

Conversely, suppose that $\lim_{k\to\infty} x_i^{(k)} = x_i$, for every $i = 1, 2, \ldots, n$.

Norms of Vectors: Distance between Vectors in $\mathbb{R}^n$

Proof (2/2)

Conversely, suppose that $\lim_{k\to\infty} x_i^{(k)} = x_i$, for every $i = 1, 2, \ldots, n$. For a given $\varepsilon > 0$, let $N_i(\varepsilon)$ for each i represent an integer with the property that

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whenever $k \geq N_i(\varepsilon)$.

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whenever $k \geq N_i(\varepsilon)$. Define $N(\varepsilon) = \max_{i=1,2,\ldots,n} N_i(\varepsilon)$.

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This implies that $\{\mathbf{x}^{(k)}\}$ converges to $\mathbf{x}$ with respect to the l_∞ norm.

Norms of Vectors: Distance between Vectors in $\mathbb{R}^n$

Example

Show that

$$\mathbf{x}^{(k)} = (x_1^{(k)}, x_2^{(k)}, x_3^{(k)}, x_4^{(k)})^t = \left(1,\ 2 + \frac{1}{k},\ \frac{3}{k^2},\ e^{-k}\sin k\right)^t$$

converges to $\mathbf{x} = (1, 2, 0, 0)^t$ with respect to the l_∞ norm.

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Solution

Because

$$\lim_{k\to\infty} 1 = 1, \quad \lim_{k\to\infty} (2 + 1/k) = 2, \quad \lim_{k\to\infty} 3/k^2 = 0, \quad \lim_{k\to\infty} e^{-k} \sin k = 0$$

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the foregoing theorem implies that the sequence $\{\mathbf{x}^{(k)}\}$ converges to $(1, 2, 0, 0)^t$ with respect to the l_∞ norm.

Norms of Vectors: Distance between Vectors in $\mathbb{R}^n$

To show directly that the sequence in the last example, namely

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Theorem

For each $\mathbf{x} \in \mathbb{R}^n$,

$$\|\mathbf{x}\|_\infty \leq \|\mathbf{x}\|_2 \leq \sqrt{n}\|\mathbf{x}\|_\infty$$

Norms of Vectors: Distance between Vectors in $\mathbb{R}^n$

Proof

Let x_j be a coordinate of $\mathbf{x}$ such that $\|\mathbf{x}\|_\infty = \max_{1\leq i\leq n} |x_i| = |x_j|$.

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Let x_j be a coordinate of $\mathbf{x}$ such that $\|\mathbf{x}\|_\infty = \max_{1 \le i \le n} |x_i| = |x_j|$. Then

$$\|\mathbf{x}\|_\infty^2 = |x_j|^2 = x_j^2 \le \sum_{i=1}^{n} x_i^2 = \|\mathbf{x}\|_2^2$$

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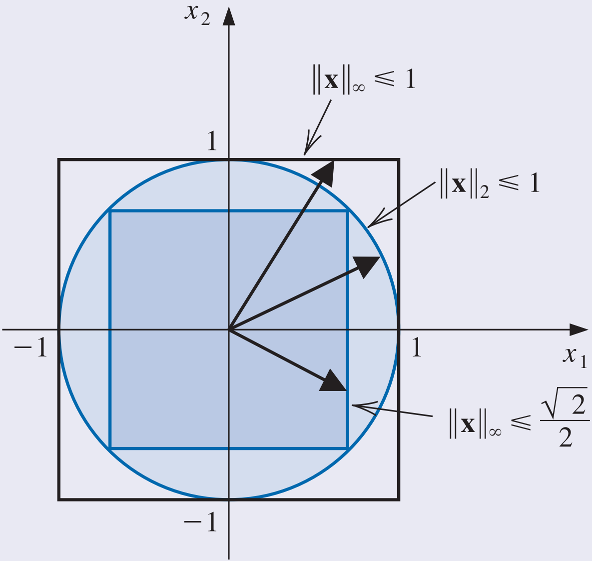
and $\|\mathbf{x}\|_2 \le \sqrt{n}\|\mathbf{x}\|_\infty$.

$\|\mathbf{x}\|_\infty \leq \|\mathbf{x}\|_2 \leq \sqrt{n}\|\mathbf{x}\|_\infty$ Illustration when $n = 2$



Norms of Vectors: Distance between Vectors in $\mathbb{R}^n$

Example

In the previous example, we found that the sequence $\{\mathbf{x}^{(k)}\}$, defined by

$$\mathbf{x}^{(k)} = \left(1, 2 + \frac{1}{k}, \frac{3}{k^2}, e^{-k}\sin k\right)^t$$

converges to $\mathbf{x} = (1, 2, 0, 0)^t$ with respect to the l_∞ norm.

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converges to $\mathbf{x} = (1, 2, 0, 0)^t$ with respect to the l_∞ norm. Show that this sequence also converges to $\mathbf{x}$ with respect to the l_2 norm.

Norms of Vectors: Distance between Vectors in $\mathbb{R}^n$

Solution

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Solution

Given any $\varepsilon > 0$, there exists an integer $N(\varepsilon/2)$ with the property that

$$\|\mathbf{x}^{(k)} - \mathbf{x}\|_\infty < \frac{\varepsilon}{2}$$

whenever $k \geq N(\varepsilon/2)$.

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Outline

1. Vector Norms
2. Distance between Vectors in $\mathbb{R}^n$
3. Matrix Norms & Distances

Matrix Norms & Distances

Definition: Matrix Norm

A matrix norm on the set of all $n \times n$ matrices is a real-valued function, $\|\cdot\|$, defined on this set, satisfying for all $n \times n$ matrices A and B and all real numbers α:

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The distance between $n \times n$ matrices A and B with respect to this matrix norm is $\|A - B\|$.

Matrix Norms & Distances

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If $||\cdot||$ is a vector norm on $\mathbb{R}^n$, then

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Matrix norms defined by vector norms are called the natural, or *induced*, matrix norm associated with the vector norm. All matrix norms will be assumed to be natural matrix norms unless specified otherwise.

Matrix Norms & Distances

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For any $\mathbf{z} \neq \mathbf{0}$, the vector

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and we can alternatively write

$$\|A\| = \max_{\mathbf{z}\neq\mathbf{0}} \frac{\|A\mathbf{z}\|}{\|\mathbf{z}\|}$$

Matrix Norms & Distances

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The following corollary to the matrix norm theorem follows from this representation of $\|A\|$.

Matrix Norms & Distances

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The following corollary to the matrix norm theorem follows from this representation of $\|A\|$.

Corollary (to the Matrix Norm Theorem)

For any vector $\mathbf{z} \neq \mathbf{0}$, matrix A, and any natural norm $\|\cdot\|$, we have

$$\|A\mathbf{z}\| \leq \|A\| \cdot \|\mathbf{z}\|$$

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The measure given to a matrix under a natural norm describes how the matrix stretches unit vectors relative to that norm. The maximum stretch is the norm of the matrix. The matrix norms we will consider have the forms

$$\|A\|_\infty = \max_{\|\mathbf{x}\|_\infty = 1} \|A\mathbf{x}\|_\infty, \quad \text{the } l_\infty \text{ norm}$$

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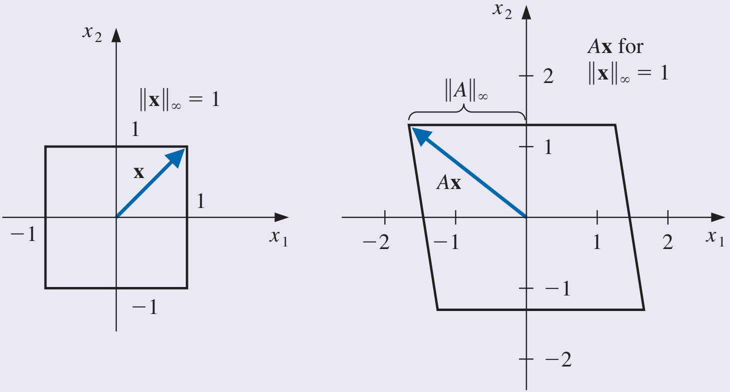
$$\|A\|_\infty = \max_{\|\mathbf{x}\|_\infty = 1} \|A\mathbf{x}\|_\infty, \quad \text{the } l_\infty \text{ norm}$$

and

$$\|A\|_2 = \max_{\|\mathbf{x}\|_2 = 1} \|A\mathbf{x}\|_2, \quad \text{the } l_2 \text{ norm}$$

An illustration of $\|A\|_\infty = \max_{\|\mathbf{x}\|_\infty=1} \|A\mathbf{x}\|_\infty$ when $n = 2$ for the matrix

$$A = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$$



An illustration of $\|A\|_2 = \max_{\|\mathbf{x}\|_2=1} \|A\mathbf{x}\|_2$ when $n = 2$ for the matrix

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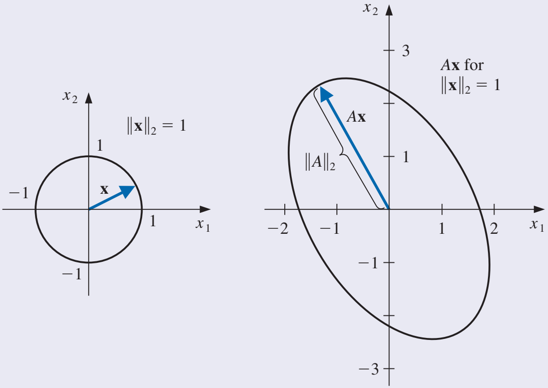

Matrix Norms & Distances

The l_∞ norm of a matrix can be easily computed from the entries of the matrix.

Matrix Norms & Distances

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Theorem

If $A = (a_{ij})$ is an $n \times n$ matrix, then

$$\|A\|_\infty = \max_{1 \le i \le n} \sum_{j=1}^{n} |a_{ij}|$$

Matrix Norms & Distances

Proof (1/4)

First we show that $\|A\|_\infty \leq \max_{1\leq i\leq n} \sum_{j=1}^{n} |a_{ij}|$.

Matrix Norms & Distances

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Matrix Norms & Distances

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$$\|A\mathbf{x}\|_\infty = \max_{1\leq i\leq n} |(A\mathbf{x})_i| = \max_{1\leq i\leq n} \left| \sum_{j=1}^{n} a_{ij}x_j \right| \leq \max_{1\leq i\leq n} \sum_{j=1}^{n} |a_{ij}| \max_{1\leq j\leq n} |x_j|$$

Matrix Norms & Distances

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But $\max_{1\leq j\leq n} |x_j| = \|\mathbf{x}\|_\infty = 1$, so

$$\|A\mathbf{x}\|_\infty \leq \max_{1\leq i\leq n} \sum_{j=1}^{n} |a_{ij}|$$

Matrix Norms & Distances

Proof (2/4)

and consequently,

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Matrix Norms & Distances

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Matrix Norms & Distances

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Matrix Norms & Distances

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$$\sum_{j=1}^{n} |a_{pj}| = \max_{1\leq i\leq n} \sum_{j=1}^{n} |a_{ij}|$$

and $\mathbf{x}$ be the vector with components

$$x_j = \begin{cases} 1, & \text{if } a_{pj} \geq 0 \\ -1, & \text{if } a_{pj} < 0 \end{cases}$$

Matrix Norms & Distances

Proof (3/4)

Then $\|\mathbf{x}\|_\infty = 1$ and $a_{pj}x_j = |a_{pj}|$, for all $j = 1, 2, \ldots, n$,

Matrix Norms & Distances

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Matrix Norms & Distances

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This result implies that

$$\|A\|_\infty = \max_{\|\mathbf{x}\|_\infty=1} \|A\mathbf{x}\|_\infty \ge \max_{1\le i\le n} \sum_{j=1}^n |a_{ij}|$$

Matrix Norms & Distances

Proof (4/4)

Putting this together with the inequality

$$\|A\|_\infty = \max_{\|\mathbf{x}\|_\infty = 1} \|A\mathbf{x}\|_\infty \leq \max_{1 \leq i \leq n} \sum_{j=1}^{n} |a_{ij}|$$

Matrix Norms & Distances

Proof (4/4)

Putting this together with the inequality

$$\|A\|_\infty = \max_{\|\mathbf{x}\|_\infty=1} \|A\mathbf{x}\|_\infty \leq \max_{1\leq i\leq n} \sum_{j=1}^{n} |a_{ij}|$$

gives

$$\|A\|_\infty = \max_{1\leq i\leq n} \sum_{j=1}^{n} |a_{ij}|$$

Matrix Norms & Distances

Example

Determine $\|A\|_\infty$ for the matrix

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & -1 \\ 5 & -1 & 1 \end{bmatrix}$$

Matrix Norms & Distances

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Solution

We have

$$\sum_{j=1}^{3} |a_{1j}| = |1| + |2| + |-1| = 4, \quad \sum_{j=1}^{3} |a_{2j}| = |0| + |3| + |-1| = 4$$

Matrix Norms & Distances

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Matrix Norms & Distances

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So the previous theorem implies that $\|A\|_\infty = \max\{4, 4, 7\} = 7$.

Questions?

Definition: Vector Norm

A vector norm on $\mathbb{R}^n$ is a function, $\|\cdot\|$, from $\mathbb{R}^n$ into $\mathbb{R}$ with the following properties:

(i) $\|\mathbf{x}\| \geq 0$ for all $\mathbf{x} \in \mathbb{R}^n$

(ii) $\|\mathbf{x}\| = 0$ if and only if $\mathbf{x} = \mathbf{0}$

(iii) $\|\alpha\mathbf{x}\| = |\alpha|\|\mathbf{x}\|$ for all $\alpha \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^n$

(iv) $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.

Return to Establishing the Properties of a Vector Norm