Iterative Techniques in Matrix Algebra

Eigenvalues & Eigenvectors

Numerical Analysis (9th Edition)

R L Burden & J D Faires

Beamer Presentation Slides
prepared by
John Carroll
Dublin City University

Outline

Outline

Outline

Outline

Outline

Eigenvalues & Eigenvectors

Matrix-Vector Multiplication

- An $n \times m$ matrix can be considered as a function that uses matrix multiplication to take m-dimensional column vectors into n-dimensional column vectors.

Eigenvalues & Eigenvectors

Matrix-Vector Multiplication

- An $n \times m$ matrix can be considered as a function that uses matrix multiplication to take m-dimensional column vectors into n-dimensional column vectors.
- So an $n \times m$ matrix is actually a linear function from $\mathbb{R}^m$ to $\mathbb{R}^n$.

Eigenvalues & Eigenvectors

Matrix-Vector Multiplication

- An $n \times m$ matrix can be considered as a function that uses matrix multiplication to take m-dimensional column vectors into n-dimensional column vectors.
- So an $n \times m$ matrix is actually a linear function from $\mathbb{R}^m$ to $\mathbb{R}^n$.
- A square matrix A takes the set of n-dimensional vectors into itself, which gives a linear function from $\mathbb{R}^n$ to $\mathbb{R}^n$.

Eigenvalues & Eigenvectors

Matrix-Vector Multiplication

- An $n \times m$ matrix can be considered as a function that uses matrix multiplication to take m-dimensional column vectors into n-dimensional column vectors.
- So an $n \times m$ matrix is actually a linear function from $\mathbb{R}^m$ to $\mathbb{R}^n$.
- A square matrix A takes the set of n-dimensional vectors into itself, which gives a linear function from $\mathbb{R}^n$ to $\mathbb{R}^n$.
- In this case, certain nonzero vectors $\mathbf{x}$ might be parallel to $A\mathbf{x}$, which means that a constant λ exists with

$$A\mathbf{x} = \lambda\mathbf{x}$$

Eigenvalues & Eigenvectors

$$A\mathbf{x} = \lambda\mathbf{x}$$

Matrix-Vector Multiplication (Cont'd)

Eigenvalues & Eigenvectors

$$A\mathbf{x} = \lambda\mathbf{x}$$

Matrix-Vector Multiplication (Cont'd)

- For these vectors, we have

$$(A - \lambda I)\mathbf{x} = \mathbf{0}$$

Eigenvalues & Eigenvectors

$$A\mathbf{x} = \lambda \mathbf{x}$$

Matrix-Vector Multiplication (Cont'd)

- For these vectors, we have

$$(A - \lambda I)\mathbf{x} = \mathbf{0}$$

- There is a close connection between these numbers λ and the likelihood that an iterative method will converge.

Eigenvalues & Eigenvectors

$$A\mathbf{x} = \lambda\mathbf{x}$$

Matrix-Vector Multiplication (Cont'd)

- For these vectors, we have

$$(A - \lambda I)\mathbf{x} = \mathbf{0}$$

- There is a close connection between these numbers λ and the likelihood that an iterative method will converge.
- We will consider this connection in this section.

Outline

Eigenvalues & Eigenvectors

Definition: Characteristic Polynomial

If A is a square matrix, the characteristic polynomial of A is defined by

$$p(\lambda) = \det(A - \lambda I)$$

Eigenvalues & Eigenvectors

Definition: Characteristic Polynomial

If A is a square matrix, the characteristic polynomial of A is defined by

$$p(\lambda) = \det(A - \lambda I)$$

Comments

- It is not difficult to show that p is an nth-degree polynomial and, consequently, has at most n distinct zeros, some of which might be complex.

Eigenvalues & Eigenvectors

Definition: Characteristic Polynomial

If A is a square matrix, the characteristic polynomial of A is defined by

$$p(\lambda) = \det(A - \lambda I)$$

Comments

- It is not difficult to show that p is an nth-degree polynomial and, consequently, has at most n distinct zeros, some of which might be complex.
- If λ is a zero of p, then, since $\det(A - \lambda I) = 0$, we can prove that the linear system defined by

$$(A - \lambda I)\mathbf{x} = \mathbf{0}$$

has a solution with $\mathbf{x} \neq \mathbf{0}$.

Eigenvalues & Eigenvectors

Definition: Eigenvalues & Eigenvectors

- If p is the characteristic polynomial of the matrix A, the zeros of p are eigenvalues, or characteristic values, of the matrix A.

Eigenvalues & Eigenvectors

Definition: Eigenvalues & Eigenvectors

- If p is the characteristic polynomial of the matrix A, the zeros of p are eigenvalues, or characteristic values, of the matrix A.
- If λ is an eigenvalue of A and $\mathbf{x} \neq \mathbf{0}$ satisfies

$$(A - \lambda I)\mathbf{x} = \mathbf{0}$$

then $\mathbf{x}$ is an eigenvector, or characteristic vector, of A corresponding to the eigenvalue λ.

Eigenvalues & Eigenvectors

Finding the Eigenvalues & Eigenvectors

- To determine the eigenvalues of a matrix, we can use the fact that λ is an eigenvalue of A if and only if

$$\det(A - \lambda I) = 0$$

Eigenvalues & Eigenvectors

Finding the Eigenvalues & Eigenvectors

- To determine the eigenvalues of a matrix, we can use the fact that λ is an eigenvalue of A if and only if

$$\det(A - \lambda I) = 0$$

- Once an eigenvalue λ has been found, a corresponding eigenvector $\mathbf{x} \neq \mathbf{0}$ is determined by solving the system

$$(A - \lambda I)\mathbf{x} = \mathbf{0}$$

Eigenvalues & Eigenvectors

Example

Show that there are no nonzero vectors **x** in $\mathbb{R}^2$ with $A\mathbf{x}$ parallel to **x** if

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Eigenvalues & Eigenvectors

Example

Show that there are no nonzero vectors **x** in $\mathbb{R}^2$ with $A\mathbf{x}$ parallel to **x** if

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Solution (1/2)

The eigenvalues of A

Eigenvalues & Eigenvectors

Example

Show that there are no nonzero vectors **x** in $\mathbb{R}^2$ with $A\mathbf{x}$ parallel to **x** if

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Solution (1/2)

The eigenvalues of A are the solutions to the characteristic polynomial

$$0 = \det(A - \lambda I) = \det \begin{bmatrix} -\lambda & 1 \\ -1 & -\lambda \end{bmatrix} = \lambda^2 + 1$$

Eigenvalues & Eigenvectors

Example

Show that there are no nonzero vectors $\mathbf{x}$ in $\mathbb{R}^2$ with $A\mathbf{x}$ parallel to $\mathbf{x}$ if

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Solution (1/2)

The eigenvalues of A are the solutions to the characteristic polynomial

$$0 = \det(A - \lambda I) = \det \begin{bmatrix} -\lambda & 1 \\ -1 & -\lambda \end{bmatrix} = \lambda^2 + 1$$

so the eigenvalues of A are the complex numbers $\lambda_1 = i$ and $\lambda_2 = -i$.

Eigenvalues & Eigenvectors

Solution (2/2)

- A corresponding eigenvector **x** for λ_1

Eigenvalues & Eigenvectors

Solution (2/2)

- A corresponding eigenvector **x** for λ_1 needs to satisfy

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -ix_1 + x_2 \\ -x_1 - ix_2 \end{bmatrix}$$

Eigenvalues & Eigenvectors

Solution (2/2)

- A corresponding eigenvector **x** for λ_1 needs to satisfy

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -ix_1 + x_2 \\ -x_1 - ix_2 \end{bmatrix}$$

that is, $0 = -ix_1 + x_2$, so $x_2 = ix_1$, and $0 = -x_1 - ix_2$.

Eigenvalues & Eigenvectors

Solution (2/2)

- A corresponding eigenvector **x** for λ_1 needs to satisfy

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -ix_1 + x_2 \\ -x_1 - ix_2 \end{bmatrix}$$

  that is, $0 = -ix_1 + x_2$, so $x_2 = ix_1$, and $0 = -x_1 - ix_2$.

- Hence if **x** is an eigenvector of A, then exactly one of its components is real and the other is complex.

Eigenvalues & Eigenvectors

Solution (2/2)

- A corresponding eigenvector **x** for λ_1 needs to satisfy

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -ix_1 + x_2 \\ -x_1 - ix_2 \end{bmatrix}$$

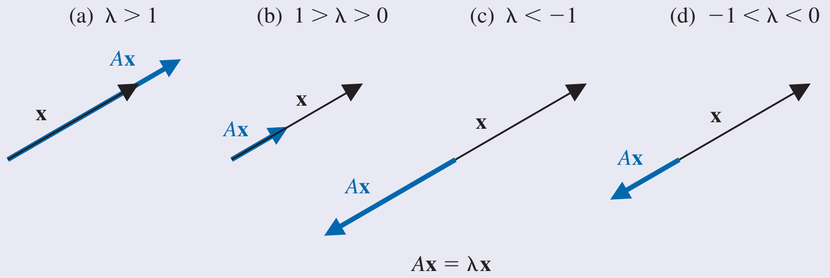
  that is, $0 = -ix_1 + x_2$, so $x_2 = ix_1$, and $0 = -x_1 - ix_2$.
- Hence if **x** is an eigenvector of A, then exactly one of its components is real and the other is complex.

As a consequence, there are no nonzero vectors **x** in $\mathbb{R}^2$ with $A\mathbf{x}$ parallel to **x**.

Eigenvalues & Eigenvectors

Geometric Interpretation of λ

- If λ is real and $\lambda > 1$, then A has the effect of stretching **x** by a factor of λ (see (a)).
- If $0 < \lambda < 1$, then A shrinks **x** by a factor of λ (see (b)).
- If $\lambda < 0$, the effects are similar (see (c) and (d)), although the direction of $A\mathbf{x}$ is reversed.



Finding the Eigenvalues & Eigenvectors of A

Example

Determine the eigenvalues and eigenvectors for the matrix

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 2 \\ 1 & -1 & 4 \end{bmatrix}$$

Finding the Eigenvalues & Eigenvectors of A

Solution (1/4)

The characteristic polynomial of A is

$$p(\lambda) = \det(A - \lambda I)$$

Finding the Eigenvalues & Eigenvectors of A

Solution (1/4)

The characteristic polynomial of A is

$$p(\lambda) = \det(A - \lambda I) = \det \begin{bmatrix} 2-\lambda & 0 & 0 \\ 1 & 1-\lambda & 2 \\ 1 & -1 & 4-\lambda \end{bmatrix}$$

Finding the Eigenvalues & Eigenvectors of A

Solution (1/4)

The characteristic polynomial of A is

$$\begin{aligned} p(\lambda) &= \det(A - \lambda I) = \det \begin{bmatrix} 2-\lambda & 0 & 0 \\ 1 & 1-\lambda & 2 \\ 1 & -1 & 4-\lambda \end{bmatrix} \\ &= -(\lambda^3 - 7\lambda^2 + 16\lambda - 12) \end{aligned}$$

Finding the Eigenvalues & Eigenvectors of A

Solution (1/4)

The characteristic polynomial of A is

$$\begin{aligned} p(\lambda) &= \det(A - \lambda I) = \det \begin{bmatrix} 2-\lambda & 0 & 0 \\ 1 & 1-\lambda & 2 \\ 1 & -1 & 4-\lambda \end{bmatrix} \\ &= -(\lambda^3 - 7\lambda^2 + 16\lambda - 12) \\ &= -(\lambda - 3)(\lambda - 2)^2 \end{aligned}$$

Finding the Eigenvalues & Eigenvectors of A

Solution (1/4)

The characteristic polynomial of A is

$$
\begin{aligned}
p(\lambda) &= \det(A - \lambda I) = \det \begin{bmatrix} 2-\lambda & 0 & 0 \\ 1 & 1-\lambda & 2 \\ 1 & -1 & 4-\lambda \end{bmatrix} \\
&= -(\lambda^3 - 7\lambda^2 + 16\lambda - 12) \\
&= -(\lambda - 3)(\lambda - 2)^2
\end{aligned}
$$

so there are two eigenvalues of A: $\lambda_1 = 3$ and $\lambda_2 = 2$.

Finding the Eigenvalues & Eigenvectors of A

Solution (2/4)

An eigenvector $\mathbf{x}_1$ corresponding to the eigenvalue $\lambda_1 = 3$ is a solution to the vector-matrix equation $(A - 3 \cdot I)\mathbf{x}_1 = \mathbf{0}$,

Finding the Eigenvalues & Eigenvectors of A

Solution (2/4)

An eigenvector $\mathbf{x}_1$ corresponding to the eigenvalue $\lambda_1 = 3$ is a solution to the vector-matrix equation $(A - 3 \cdot I)\mathbf{x}_1 = \mathbf{0}$, so

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 1 & -2 & 2 \\ 1 & -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

which implies that $x_1 = 0$ and $x_2 = x_3$.

Finding the Eigenvalues & Eigenvectors of A

Solution (2/4)

An eigenvector $\mathbf{x}_1$ corresponding to the eigenvalue $\lambda_1 = 3$ is a solution to the vector-matrix equation $(A - 3 \cdot I)\mathbf{x}_1 = \mathbf{0}$, so

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 1 & -2 & 2 \\ 1 & -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

which implies that $x_1 = 0$ and $x_2 = x_3$.

Any nonzero value of x_3 produces an eigenvector for the eigenvalue $\lambda_1 = 3$.

Finding the Eigenvalues & Eigenvectors of A

Solution (2/4)

An eigenvector $\mathbf{x}_1$ corresponding to the eigenvalue $\lambda_1 = 3$ is a solution to the vector-matrix equation $(A - 3 \cdot I)\mathbf{x}_1 = \mathbf{0}$, so

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 1 & -2 & 2 \\ 1 & -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

which implies that $x_1 = 0$ and $x_2 = x_3$.

Any nonzero value of x_3 produces an eigenvector for the eigenvalue $\lambda_1 = 3$. For example, when $x_3 = 1$ we have the eigenvector $\mathbf{x}_1 = (0, 1, 1)^t$, and any eigenvector of A corresponding to $\lambda = 3$ is a nonzero multiple of $\mathbf{x}_1$.

Finding the Eigenvalues & Eigenvectors of A

Solution (3/4)

An eigenvector $\mathbf{x} \neq \mathbf{0}$ of A associated with $\lambda_2 = 2$ is a solution of the system $(A - 2 \cdot I)\mathbf{x} = \mathbf{0}$,

Finding the Eigenvalues & Eigenvectors of A

Solution (3/4)

An eigenvector $\mathbf{x} \neq \mathbf{0}$ of A associated with $\lambda_2 = 2$ is a solution of the system $(A - 2 \cdot I)\mathbf{x} = \mathbf{0}$, so

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & -1 & 2 \\ 1 & -1 & 2 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Finding the Eigenvalues & Eigenvectors of A

Solution (3/4)

An eigenvector $\mathbf{x} \neq \mathbf{0}$ of A associated with $\lambda_2 = 2$ is a solution of the system $(A - 2 \cdot I)\mathbf{x} = \mathbf{0}$, so

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & -1 & 2 \\ 1 & -1 & 2 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

In this case the eigenvector has only to satisfy the equation

$$x_1 - x_2 + 2x_3 = 0$$

which can be done in various ways.

Finding the Eigenvalues & Eigenvectors of A

Solution (4/4)

- For example, when $x_1 = 0$ we have $x_2 = 2x_3$, so one choice would be $\mathbf{x}_2 = (0, 2, 1)^t$.

Finding the Eigenvalues & Eigenvectors of A

Solution (4/4)

- For example, when $x_1 = 0$ we have $x_2 = 2x_3$, so one choice would be $\mathbf{x}_2 = (0, 2, 1)^t$.
- We could also choose $x_2 = 0$, which requires that $x_1 = -2x_3$.

Finding the Eigenvalues & Eigenvectors of A

Solution (4/4)

- For example, when $x_1 = 0$ we have $x_2 = 2x_3$, so one choice would be $\mathbf{x}_2 = (0, 2, 1)^t$.
- We could also choose $x_2 = 0$, which requires that $x_1 = -2x_3$.
- Hence $\mathbf{x}_3 = (-2, 0, 1)^t$ gives a second eigenvector for the eigenvalue $\lambda_2 = 2$ that is not a multiple of $\mathbf{x}_2$.

Finding the Eigenvalues & Eigenvectors of A

Solution (4/4)

- For example, when $x_1 = 0$ we have $x_2 = 2x_3$, so one choice would be $\mathbf{x}_2 = (0, 2, 1)^t$.
- We could also choose $x_2 = 0$, which requires that $x_1 = -2x_3$.
- Hence $\mathbf{x}_3 = (-2, 0, 1)^t$ gives a second eigenvector for the eigenvalue $\lambda_2 = 2$ that is not a multiple of $\mathbf{x}_2$.
- The eigenvectors of A corresponding to the eigenvalue $\lambda_2 = 2$ generate an entire plane. This plane is described by all vectors of the form

$$\alpha \mathbf{x}_2 + \beta \mathbf{x}_3 = (-2\beta, 2\alpha, \alpha + \beta)^t$$

  for arbitrary constants α and β, provided that at least one of the constants is nonzero.

Outline

Eigenvalues & Eigenvectors: Spectral Radius

Definition: Spectral Radius

The spectral radius $\rho(A)$ of a matrix A is defined by

$$\rho(A) = \max |\lambda|, \quad \text{where } \lambda \text{ is an eigenvalue of } A$$

Eigenvalues & Eigenvectors: Spectral Radius

Definition: Spectral Radius

The spectral radius $\rho(A)$ of a matrix A is defined by

$$\rho(A) = \max |\lambda|, \quad \text{where } \lambda \text{ is an eigenvalue of } A$$

(For complex $\lambda = \alpha + \beta i$, we define $|\lambda| = (\alpha^2 + \beta^2)^{1/2}$.)

Eigenvalues & Eigenvectors: Spectral Radius

Definition: Spectral Radius

The spectral radius $\rho(A)$ of a matrix A is defined by

$$\rho(A) = \max |\lambda|, \qquad \text{where } \lambda \text{ is an eigenvalue of } A$$

(For complex $\lambda = \alpha + \beta i$, we define $|\lambda| = (\alpha^2 + \beta^2)^{1/2}$.)

For the matrix in the previous example, namely

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 2 \\ 1 & -1 & 4 \end{bmatrix}$$

note that

$$\rho(A) = \max\{2, 3\} = 3$$

Eigenvalues & Eigenvectors: Spectral Radius

The spectral radius is closely related to the norm of a matrix, as shown in the following theorem.

Eigenvalues & Eigenvectors: Spectral Radius

The spectral radius is closely related to the norm of a matrix, as shown in the following theorem.

Theorem

If A is an $n \times n$ matrix,

Eigenvalues & Eigenvectors: Spectral Radius

The spectral radius is closely related to the norm of a matrix, as shown in the following theorem.

Theorem

If A is an $n \times n$ matrix, then

(i) $\|A\|_2 = [\rho(A^t A)]^{1/2}$

Eigenvalues & Eigenvectors: Spectral Radius

The spectral radius is closely related to the norm of a matrix, as shown in the following theorem.

Theorem

If A is an $n \times n$ matrix, then

(i) $\|A\|_2 = [\rho(A^t A)]^{1/2}$

(ii) $\rho(A) \leq \|A\|$

Eigenvalues & Eigenvectors: Spectral Radius

The spectral radius is closely related to the norm of a matrix, as shown in the following theorem.

Theorem

If A is an $n \times n$ matrix, then

(i) $\|A\|_2 = [\rho(A^t A)]^{1/2}$

(ii) $\rho(A) \leq \|A\|$

for any natural norm $\|\cdot\|$

Eigenvalues & Eigenvectors: Spectral Radius

$$\|A\|_2 = [\rho(A^t A)]^{1/2}$$

Proof (1/2)

Eigenvalues & Eigenvectors: Spectral Radius

$$\|A\|_2 = [\rho(A^t A)]^{1/2}$$

Proof (1/2)

- The proof of part (i) requires more information concerning eigenvalues than we presently have available.

Eigenvalues & Eigenvectors: Spectral Radius

$$\|A\|_2 = [\rho(A^t A)]^{1/2}$$

Proof (1/2)

- The proof of part (i) requires more information concerning eigenvalues than we presently have available.
- For the details involved in the proof, see p. 21 of Ortega, J. M., Numerical Analysis; a Second Course, Academic Press, New York, 1972, 201 pp.

Eigenvalues & Eigenvectors: Spectral Radius

$$\|A\|_2 = [\rho(A^t A)]^{1/2}$$

Proof (1/2)

- The proof of part (i) requires more information concerning eigenvalues than we presently have available.
- For the details involved in the proof, see p. 21 of Ortega, J. M., Numerical Analysis; a Second Course, Academic Press, New York, 1972, 201 pp.

Note: It can be shown that this result implies that if A is symmetric, then $\|A\|_2 = \rho(A)$.

Eigenvalues & Eigenvectors: Spectral Radius

$$\rho(A) \leq \|A\|$$

Eigenvalues & Eigenvectors: Spectral Radius

$$\rho(A) \leq \|A\|$$

Proof (2/2)

To prove part (ii), suppose λ is an eigenvalue of A with eigenvector $\mathbf{x}$ and $\|\mathbf{x}\| = 1$.

Eigenvalues & Eigenvectors: Spectral Radius

$$\rho(A) \leq \|A\|$$

Proof (2/2)

To prove part (ii), suppose λ is an eigenvalue of A with eigenvector $\mathbf{x}$ and $\|\mathbf{x}\| = 1$. Then $A\mathbf{x} = \lambda\mathbf{x}$ and

$$|\lambda| = |\lambda| \cdot \|\mathbf{x}\| = \|\lambda\mathbf{x}\| = \|A\mathbf{x}\| \leq \|A\| \|\mathbf{x}\| = \|A\|$$

Eigenvalues & Eigenvectors: Spectral Radius

$$\rho(A) \leq \|A\|$$

Proof (2/2)

To prove part (ii), suppose λ is an eigenvalue of A with eigenvector $\mathbf{x}$ and $\|\mathbf{x}\| = 1$. Then $A\mathbf{x} = \lambda\mathbf{x}$ and

$$|\lambda| = |\lambda| \cdot \|\mathbf{x}\| = \|\lambda\mathbf{x}\| = \|A\mathbf{x}\| \leq \|A\| \|\mathbf{x}\| = \|A\|$$

Thus

$$\rho(A) = \max |\lambda| \leq \|A\|$$

Eigenvalues & Eigenvectors: Spectral Radius

Example

Determine the l_2 norm of

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix}$$

Eigenvalues & Eigenvectors: Spectral Radius

Example

Determine the l_2 norm of

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix}$$

Note: We will apply part (i) of the theorem, namely that

$$\|A\|_2 = [\rho(A^t A)]^{1/2}$$

Eigenvalues & Eigenvectors: Spectral Radius

$$\|A\|_2 = [\rho(A^t A)]^{1/2}$$

Solution (1/3)

We first need the eigenvalues of $A^t A$, where

$$A^t A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 2 & -1 \\ 2 & 6 & 4 \\ -1 & 4 & 5 \end{bmatrix}$$

Eigenvalues & Eigenvectors: Spectral Radius

$$\|A\|_2 = [\rho(A^t A)]^{1/2}$$

Solution (2/3)

If

$$0 = \det(A^t A - \lambda I)$$

Eigenvalues & Eigenvectors: Spectral Radius

$$\|A\|_2 = [\rho(A^t A)]^{1/2}$$

Solution (2/3)

If

$$0 = \det(A^t A - \lambda I) = \det \begin{bmatrix} 3-\lambda & 2 & -1 \\ 2 & 6-\lambda & 4 \\ -1 & 4 & 5-\lambda \end{bmatrix}$$

Eigenvalues & Eigenvectors: Spectral Radius

$$\|A\|_2 = [\rho(A^t A)]^{1/2}$$

Solution (2/3)

If

$$\begin{aligned} 0 = \det(A^t A - \lambda I) &= \det \begin{bmatrix} 3-\lambda & 2 & -1 \\ 2 & 6-\lambda & 4 \\ -1 & 4 & 5-\lambda \end{bmatrix} \\ &= -\lambda^3 + 14\lambda^2 - 42\lambda \end{aligned}$$

Eigenvalues & Eigenvectors: Spectral Radius

$$\|A\|_2 = [\rho(A^t A)]^{1/2}$$

Solution (2/3)

If

$$\begin{aligned} 0 = \det(A^t A - \lambda I) &= \det \begin{bmatrix} 3-\lambda & 2 & -1 \\ 2 & 6-\lambda & 4 \\ -1 & 4 & 5-\lambda \end{bmatrix} \\ &= -\lambda^3 + 14\lambda^2 - 42\lambda \\ &= -\lambda(\lambda^2 - 14\lambda + 42) \end{aligned}$$

Eigenvalues & Eigenvectors: Spectral Radius

$$\|A\|_2 = [\rho(A^t A)]^{1/2}$$

Solution (2/3)

If

$$\begin{aligned}
0 = \det(A^t A - \lambda I) &= \det \begin{bmatrix} 3-\lambda & 2 & -1 \\ 2 & 6-\lambda & 4 \\ -1 & 4 & 5-\lambda \end{bmatrix} \\
&= -\lambda^3 + 14\lambda^2 - 42\lambda \\
&= -\lambda(\lambda^2 - 14\lambda + 42)
\end{aligned}$$

then $\lambda = 0$ or $\lambda = 7 \pm \sqrt{7}$.

Eigenvalues & Eigenvectors: Spectral Radius

$$||A||_2 = [\rho(A^t A)]^{1/2}$$

Solution (3/3)

By part (i) of the theorem, we have

$$||A||_2 = \sqrt{\rho(A^t A)}$$

Eigenvalues & Eigenvectors: Spectral Radius

$$\|A\|_2 = [\rho(A^t A)]^{1/2}$$

Solution (3/3)

By part (i) of the theorem, we have

$$\begin{aligned} ||A||_2 &= \sqrt{\rho(A^t A)} \\ &= \sqrt{\max\{0,\ 7-\sqrt{7},\ 7+\sqrt{7}\}} \end{aligned}$$

Eigenvalues & Eigenvectors: Spectral Radius

$$\|A\|_2 = [\rho(A^t A)]^{1/2}$$

Solution (3/3)

By part (i) of the theorem, we have

$$\begin{aligned} ||A||_2 &= \sqrt{\rho(A^t A)} \\ &= \sqrt{\max\{0,\ 7-\sqrt{7},\ 7+\sqrt{7}\}} \\ &= \sqrt{7+\sqrt{7}} \end{aligned}$$

Eigenvalues & Eigenvectors: Spectral Radius

$$||A||_2 = [\rho(A^t A)]^{1/2}$$

Solution (3/3)

By part (i) of the theorem, we have

$$\begin{aligned}
||A||_2 &= \sqrt{\rho(A^t A)} \\
&= \sqrt{\max\{0,\ 7-\sqrt{7},\ 7+\sqrt{7}\}} \\
&= \sqrt{7+\sqrt{7}} \\
&\approx 3.106
\end{aligned}$$

Outline

Eigenvalues & Eigenvectors: Convergent Matrices

In studying iterative matrix techniques, it is of particular importance to know when powers of a matrix become small (that is, when all the entries approach zero). Matrices of this type are called convergent.

Eigenvalues & Eigenvectors: Convergent Matrices

In studying iterative matrix techniques, it is of particular importance to know when powers of a matrix become small (that is, when all the entries approach zero). Matrices of this type are called convergent.

Definition: Convergent Matrix

We call an $n \times n$ matrix A convergent if

$$\lim_{k \to \infty} (A^k)_{ij} = 0, \quad \text{for each } i = 1, 2, \ldots, n \text{ and } j = 1, 2, \ldots, n$$

Eigenvalues & Eigenvectors: Convergent Matrices

Example

Show that

$$A = \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{2} \end{bmatrix}$$

is a convergent matrix.

Eigenvalues & Eigenvectors: Convergent Matrices

Solution

Computing powers of A, we obtain:

$$A^2 = \begin{bmatrix} \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix} \qquad A^3 = \begin{bmatrix} \frac{1}{8} & 0 \\ \frac{3}{16} & \frac{1}{8} \end{bmatrix} \qquad A^4 = \begin{bmatrix} \frac{1}{16} & 0 \\ \frac{1}{8} & \frac{1}{16} \end{bmatrix}$$

Eigenvalues & Eigenvectors: Convergent Matrices

Solution

Computing powers of A, we obtain:

$$A^2 = \begin{bmatrix} \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix} \qquad A^3 = \begin{bmatrix} \frac{1}{8} & 0 \\ \frac{3}{16} & \frac{1}{8} \end{bmatrix} \qquad A^4 = \begin{bmatrix} \frac{1}{16} & 0 \\ \frac{1}{8} & \frac{1}{16} \end{bmatrix}$$

and, in general,

$$A^k = \begin{bmatrix} (\frac{1}{2})^k & 0 \\ \frac{k}{2^{k+1}} & (\frac{1}{2})^k \end{bmatrix}$$

Eigenvalues & Eigenvectors: Convergent Matrices

Solution

Computing powers of A, we obtain:

$$A^2 = \begin{bmatrix} \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix} \qquad A^3 = \begin{bmatrix} \frac{1}{8} & 0 \\ \frac{3}{16} & \frac{1}{8} \end{bmatrix} \qquad A^4 = \begin{bmatrix} \frac{1}{16} & 0 \\ \frac{1}{8} & \frac{1}{16} \end{bmatrix}$$

and, in general,

$$A^k = \begin{bmatrix} (\frac{1}{2})^k & 0 \\ \frac{k}{2^{k+1}} & (\frac{1}{2})^k \end{bmatrix}$$

So A is a convergent matrix because

$$\lim_{k\to\infty} \left(\frac{1}{2}\right)^k = 0 \quad \text{and} \quad \lim_{k\to\infty} \frac{k}{2^{k+1}} = 0$$

Eigenvalues & Eigenvectors: Convergent Matrices

- Notice that the convergent matrix A in the last example has $\rho(A) = \frac{1}{2}$, because $\frac{1}{2}$ is the only eigenvalue of A.

Eigenvalues & Eigenvectors: Convergent Matrices

- Notice that the convergent matrix A in the last example has $\rho(A) = \frac{1}{2}$, because $\frac{1}{2}$ is the only eigenvalue of A.
- This illustrates an important connection that exists between the spectral radius of a matrix and the convergence of the matrix,as detailed in the following result.

Eigenvalues & Eigenvectors: Convergent Matrices

Theorem

The following statements are equivalent.

Eigenvalues & Eigenvectors: Convergent Matrices

Theorem

The following statements are equivalent.

(i) A is a convergent matrix.

Eigenvalues & Eigenvectors: Convergent Matrices

Theorem

The following statements are equivalent.

(i) A is a convergent matrix.

(ii) $\lim_{n\to\infty} \|A^n\| = 0$, for some natural norm.

Eigenvalues & Eigenvectors: Convergent Matrices

Theorem

The following statements are equivalent.

(i) A is a convergent matrix.

(ii) $\lim_{n\to\infty} \|A^n\| = 0$, for some natural norm.

(iii) $\lim_{n\to\infty} \|A^n\| = 0$, for all natural norms.

Eigenvalues & Eigenvectors: Convergent Matrices

Theorem

The following statements are equivalent.

(i) A is a convergent matrix.

(ii) $\lim_{n\to\infty} \|A^n\| = 0$, for some natural norm.

(iii) $\lim_{n\to\infty} \|A^n\| = 0$, for all natural norms.

(iv) $\rho(A) < 1$.

Eigenvalues & Eigenvectors: Convergent Matrices

Theorem

The following statements are equivalent.

(i) A is a convergent matrix.

(ii) $\lim_{n\to\infty} \|A^n\| = 0$, for some natural norm.

(iii) $\lim_{n\to\infty} \|A^n\| = 0$, for all natural norms.

(iv) $\rho(A) < 1$.

(v) $\lim_{n\to\infty} A^n\mathbf{x} = \mathbf{0}$, for every $\mathbf{x}$.

Eigenvalues & Eigenvectors: Convergent Matrices

Theorem

The following statements are equivalent.

(i) A is a convergent matrix.

(ii) $\lim_{n\to\infty} \|A^n\| = 0$, for some natural norm.

(iii) $\lim_{n\to\infty} \|A^n\| = 0$, for all natural norms.

(iv) $\rho(A) < 1$.

(v) $\lim_{n\to\infty} A^n\mathbf{x} = \mathbf{0}$, for every $\mathbf{x}$.

The proof of this theorem can be found on p. 14 of Issacson, E. and H. B. Keller, Analysis of Numerical Methods, John Wiley & Sons, New York, 1966, 541 pp.

Questions?