Numerical Analysis

10th ed

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Beamer Presentation Slides
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Definition (1.1)

A function f defined on a set X of real numbers has the **limit** L at x_0, written

$$\lim_{x \to x_0} f(x) = L,$$

if, given any real number $\varepsilon > 0$, there exists a real number $\delta > 0$ such that

$$|f(x) - L| < \varepsilon, \quad \text{whenever} \quad x \in X \quad \text{and} \quad 0 < |x - x_0| < \delta.$$

Definition (1.2)

Let f be a function defined on a set X of real numbers and $x_0 \in X$. Then f is **continuous** at x_0 if

$$\lim_{x \to x_0} f(x) = f(x_0).$$

The function f is **continuous on the set** X if it is continuous at each number in X.

Definition (1.3)

Let $\{x_n\}_{n=1}^{\infty}$ be an infinite sequence of real numbers. This sequence has the **limit** x (**converges to** x) if, for any $\varepsilon > 0$ there exists a positive integer $N(\varepsilon)$ such that $|x_n - x| < \varepsilon$, whenever $n > N(\varepsilon)$. The notation

$$\lim_{n\to\infty} x_n = x, \quad \text{or} \quad x_n \to x \quad \text{as} \quad n \to \infty,$$

means that the sequence $\{x_n\}_{n=1}^{\infty}$ converges to x.

Theorem (1.4)

If f is a function defined on a set X of real numbers and $x_0 \in X$, then the following statements are equivalent:

a. *f is continuous at x_0;*

b. *If $\{x_n\}_{n=1}^{\infty}$ is any sequence in X converging to x_0, then $\lim_{n\to\infty} f(x_n) = f(x_0)$.*

Definition (1.5)

Let f be a function defined in an open interval containing x_0. The function f is **differentiable** at x_0 if

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists. The number $f'(x_0)$ is called the **derivative** of f at x_0. A function that has a derivative at each number in a set X is **differentiable on** X .

Theorem (1.6)

If the function f is differentiable at x_0, then f is continuous at x_0.

Theorem (1.7 Rolle's Theorem)

Suppose $f \in C[a, b]$ and f is differentiable on (a, b). If $f(a) = f(b)$, then a number c in (a, b) exists with $f'(c) = 0$. (See Figure 1.3.)

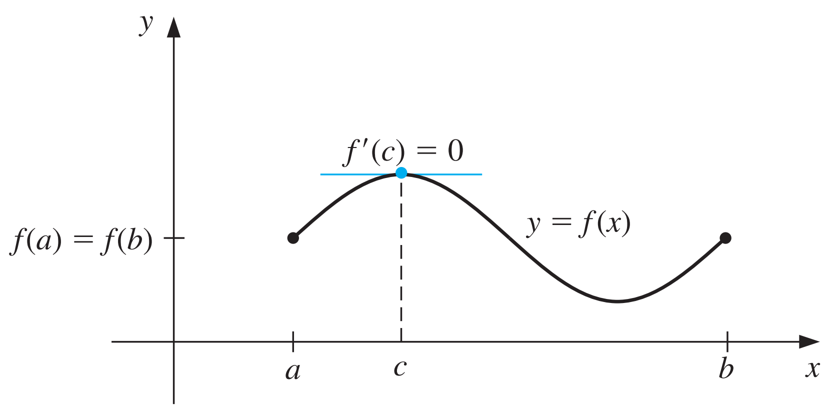

Figure: Figure 1.3

Theorem (1.8 Mean Value Theorem)

If $f \in C[a,b]$ and f is differentiable on (a,b), then a number c in (a,b) exists with (See Figure 1.4.)

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

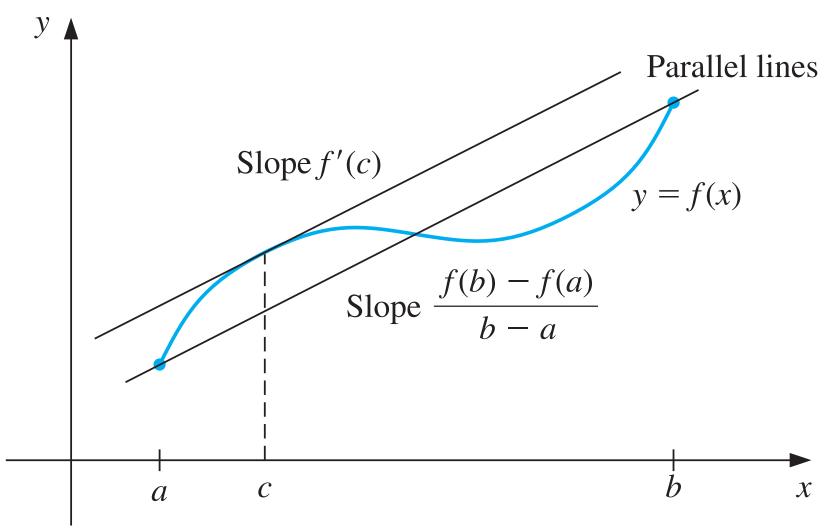

Figure: Figure 1.4

Theorem (1.9 Extreme Value Theorem)

If $f \in C[a,b]$, then c_1, $c_2 \in [a,b]$ exist with $f(c_1) \le f(x) \le f(c_2)$, for all $x \in [a,b]$. In addition, if f is differentiable on (a,b), then the numbers c_1 and c_2 occur either at the endpoints of $[a,b]$ or where f' is zero. (See Figure 1.5.)

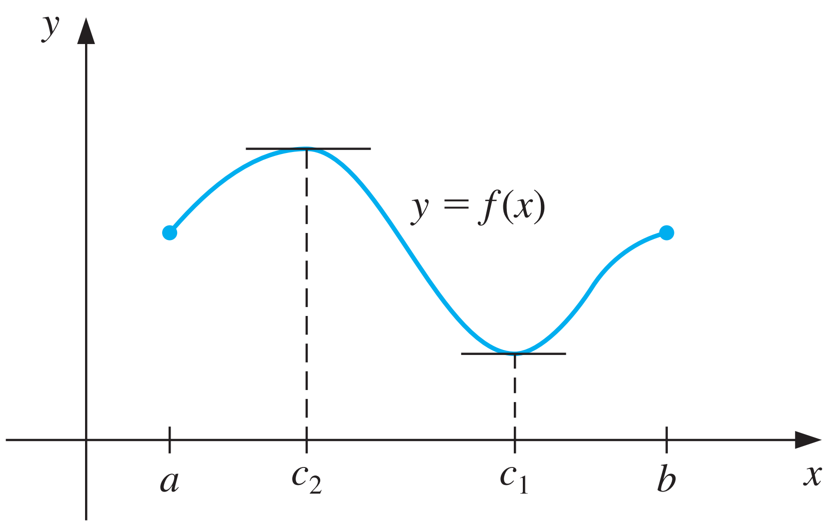

Figure: Figure 1.5

Example 1

Use the Extreme Value Theorem to find
$max_{0.1 \leq x \leq 0.6} |5x^3 + 2x^2 - 4x|$.

Solution

Since $f(x) = 5x^3 + 2x^2 - 4x$ is a polynomial function it is continuous for all $x \in [0, 0.7]$. Locate the critical points in [0.1,0.6] by finding the first derivative and setting it equal to 0.
$f'(x) = 15x^2 + 4x - 4 = 0 \quad \text{at} \quad x = 0.4$
Evaluate $|f(x)|$ at each critical point in the interval as well as at the endpoints of the interval. We have
$f(0.2) = 0.375, \quad f(0.4) = 0.960, \quad f(0.6) = 0.600$

Thus, the max occurs at (0.4,0.960).

Theorem (1.10 Generalized Rolle's Theorem)

Suppose $f \in C[a, b]$ is n times differentiable on (a, b). If $f(x) = 0$ at the $n + 1$ distinct numbers $a \leq x_0 < x_1 < \ldots < x_n \leq b$, then a number c in (x_0, x_n), and hence in (a, b), exists with $f^{(n)}(c) = 0$.

Theorem (1.11 Intermediate Value Theorem)

If $f \in C[a, b]$ and K is any number between $f(a)$ and $f(b)$, then there exists a number c in (a, b) for which $f(c) = K$.

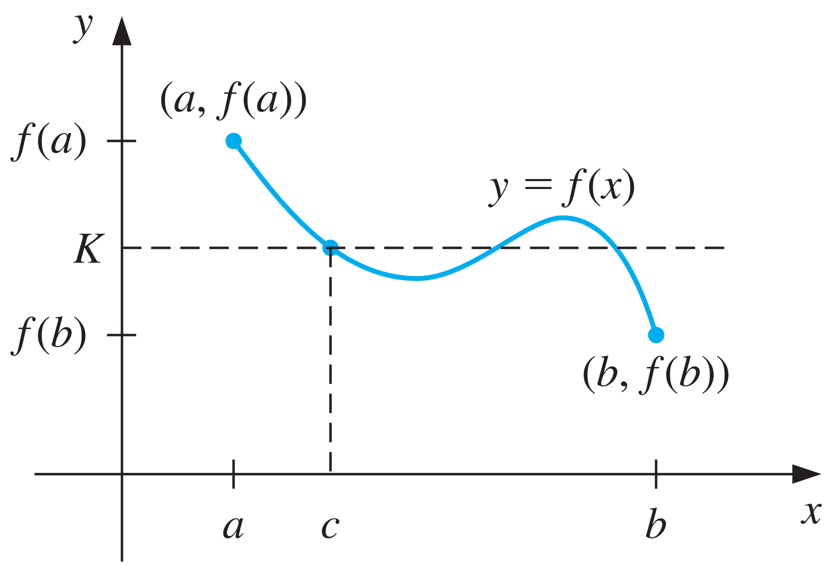

Figure: Figure 1.6

Example 2

Use the Intermediate Value Theorem to show that $5x^3 + 2x^2 - 4x = 0$; has at least one solution in a given interval $[-1, 0.5]$.

Solution

Let $f(x) = 5x^3 + 2x^2 - 4x$. Calculate the function values at each endpoint of the interval.

$$f(-1) = 1 \quad f(0.5) = -8.75$$

Since the function values are of opposite sign, there is at least one solution in the given interval.

Theorem (1.13 Weighted Mean Value Theorem for Integrals)

Suppose $f \in C[a, b]$, the Riemann integral of g exists on $[a, b]$, and $g(x)$ does not change sign on $[a, b]$. Then there exists a number c in (a, b) with

$$\int_a^b f(x)g(x)\,dx = f(c)\int_a^b g(x)\,dx.$$

NOTE: When $g(x) \equiv 1$, Theorem 1.13 is the usual Mean Value Theorem for Integrals. It gives the **average value** of the function f over the interval $[a, b]$ as (See Figure 1.8.)

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

Theorem (1.14 Taylor's Theorem)

Suppose $f \in C^n[a,b]$, that $f^{(n+1)}$ exists on $[a,b]$, and $x_0 \in [a,b]$. For every $x \in [a,b]$, there exists a number $\xi(x)$ between x_0 and x with

$$f(x) = P_n(x) + R_n(x),$$

$$\begin{aligned} P_n(x) =& f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots \\ &+ \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k \\ R_n(x) =& \frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x - x_0)^{n+1}. \end{aligned}$$

Example 3

Let $f(x) = \cos x$ and $x_0 = 0$. Determine The second Taylor polynomial for f about x_0.

Solution

Since $f \in C^\infty(\mathbb{R})$,Taylor's Theorem can be applied for any $n \geq 0$. Also, $f'(x) = -\sin x,\ f''(x) = -\cos x,\ f'''(x) = \sin x$ and $f^{(4)}(x) = \cos x$. So $f(0) = 1,\ f'(0) = 0,\ f''(0) = -1$, and $f'''(0) = 0$.

For $n = 2$ and $x_0 = 0$,

$$\cos x = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(\xi(x))}{3!}x^3$$
$$= 1 - \frac{1}{2}x^2 + \frac{1}{6}x^3 \sin \xi(x),$$

where $\xi(x)$ is some (generally unknown) number between 0 and x.

Definition (1.15)

Suppose that p^* is an approximation to p. The **actual error** is $p - p^*$, the **absolute error** is $|p - p^*|$, and the **relative error** is $\frac{|p - p^*|}{|p|}$, provided that $p \neq 0$.

Definition (1.16)

The number p^* is said to approximate p to t **significant digits** (or figures) if t is the largest nonnegative integer for which

$$\frac{|p - p^*|}{|p|} \leq 5 \times 10^{-t}.$$

Definition (1.17)

Suppose that $E_0 > 0$ denotes an error introduced at some stage in the calculations and E_n represents the magnitude of the error after n subsequent operations.

- If $E_n \approx CnE_0$, where C is a constant independent of n, then the growth of error is said to be **linear**.
- If $E_n \approx C^n E_0$, for some $C > 1$, then the growth of error is called **exponential**.

Definition (1.18)

Suppose $\{\beta_n\}_{n=1}^{\infty}$ is a sequence known to converge to zero, and $\{\alpha_n\}_{n=1}^{\infty}$ converges to a number α. If a positive constant K exists with

$$|\alpha_n - \alpha| \le K|\beta_n|, \quad \text{for large } n,$$

then we say that $\{\alpha_n\}_{n=1}^{\infty}$ converges to α with **rate, or order, of convergence** $O(\beta_n)$. (This expression is read “big oh of β_n" .) It is indicated by writing $\alpha_n = \alpha + O(\beta_n)$.