Numerical Analysis

10th ed

R L Burden, J D Faires, and A M Burden

Beamer Presentation Slides
Prepared by
Dr. Annette M. Burden
Youngstown State University

June 21, 2015

Algorithm 2.1: BISECTION

To find a solution to $f(x) = 0$ given the continuous function f on the interval $[a, b]$, where $f(a)$ and $f(b)$ have opposite signs:

INPUT endpoints a, b; tolerance TOL; maximum number of iterations N_0.
OUTPUT approximate solution p or message of failure.
Step 1 Set $i = 1$;
$FA = f(a)$.
Step 2 While $i \leq N_0$ do Steps 3–6.
Step 3 Set $p = a + (b - a)/2$; (*Compute* p_i.)
$FP = f(p)$.
Step 4 If $FP = 0$ or $(b - a)/2 < TOL$ then
OUTPUT (p); (*Procedure completed successfully.*)
STOP.
Step 5 Set $i = i + 1$.
Step 6 If $FA \cdot FP > 0$ then set $a = p$; (*Compute* a_i, b_i.)
$FA = FP$
else set $b = p$. (*FA is unchanged.*)
Step 7 OUTPUT ('Method failed after N_0 iterations, N_0 =', N_0);
(*The procedure was unsuccessful.*)
STOP.

Bisection Illustration

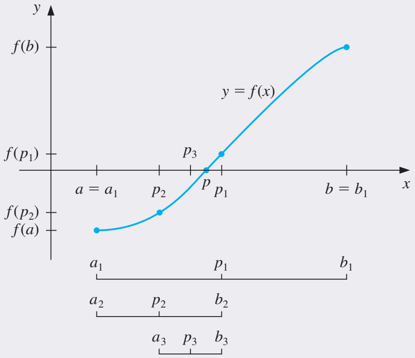

Figure: Figure 2.1

This YouTube video can serve as a good illustration of the Bisection Method for students. Bisection Video

Theorem (2.1)

Suppose that $f \in C[a, b]$ *and* $f(a) \cdot f(b) < 0$. *The Bisection method generates a sequence* $\{p_n\}_{n=1}^{\infty}$ *approximating a zero* p *of* f *with*

$$|p_n - p| \leq \frac{b-a}{2^n}, \quad \text{when} \quad n \geq 1.$$

Definition (2.2)

The number p is a **fixed point** for a given function g if $g(p) = p$.

NOTES:

- Given a root-finding problem $f(p) = 0$, we can define functions g with a fixed point at p in a number of ways, for example, as

$$g(x) = x - f(x) \quad \text{or as} \quad g(x) = x + 3f(x).$$

- Conversely, if the function g has a fixed point at p, then the function defined by

$$f(x) = x - g(x)$$

has a zero at p.

Theorem (2.3)

(i) *If $g \in C[a, b]$ and $g(x) \in [a, b]$ for all $x \in [a, b]$, then g has at least one fixed point in $[a, b]$.*

(ii) *If, in addition, $g'(x)$ exists on (a, b) and a positive constant $k < 1$ exists with $|g'(x)| \leq k$, for all $x \in (a, b)$, then there is exactly one fixed point in $[a, b]$. (See Figure 2.3.)* □

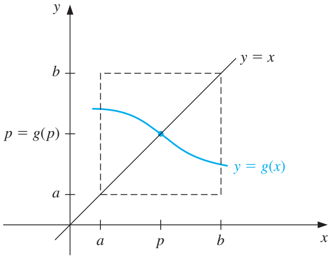

Figure: Figure 2.3

Algorithm 2.2: FIXED-POINT ITERATION

To find a solution to $p = g(p)$ given an initial approximation p_0:

INPUT initial approximation p_0; tolerance TOL; max # of iterations N_0.
OUTPUT approximate solution p or message of failure.
Step 1 Set $i = 1$.
Step 2 While $i \leq N_0$ do Steps 3–6.
Step 3 Set $p = g(p_0)$. (*Compute p_i.*)
Step 4 If $|p - p_0| < TOL$ then
OUTPUT (p); (*The procedure was successful.*)
STOP.
Step 5 Set $i = i + 1$.
Step 6 Set $p_0 = p$. (*Update p_0.*)
Step 7 OUTPUT ('The method failed after N_0 iterations, $N_0 =$', N_0);
(*The procedure was unsuccessful.*)
STOP.

Fixed-Point Illustration

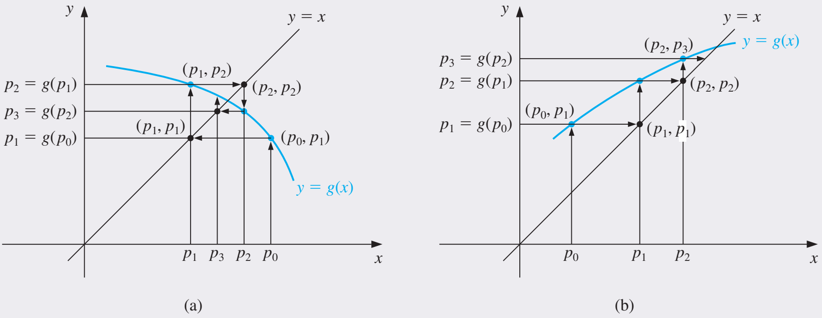

Figure: Figure 2.6

This YouTube video developed by Oscar Veliz can serve as a good illustration of the Fixed-Point Method for students.

Fixed-Point Video

Theorem (2.4: Fixed-Point Theorem)

Let $g \in C[a,b]$ be such that $g(x) \in [a,b]$, for all x in $[a,b]$. Suppose, in addition, that g' exists on (a,b) and that a constant $0 < k < 1$ exists with

$$|g'(x)| \leq k, \quad \text{for all } x \in (a,b).$$

Then for any number p_0 in $[a,b]$, the sequence defined by

$$p_n = g(p_{n-1}), \quad n \geq 1,$$

converges to the unique fixed point p in $[a,b]$.

Corollary (2.5)

If g satisfies the hypotheses of Theorem 2.4, then bounds for the error involved in using p_n to approximate p are given by

$$|p_n - p| \leq k^n \max\{p_0 - a, b - p_0\} \quad (1)$$

and

$$|p_n - p| \leq \frac{k^n}{1-k}|p_1 - p_0|, \quad \text{for all} \quad n \geq 1. \quad (2)$$

□

Algorithm 2.3: NEWTON'S METHOD

To find a solution to $f(x) = 0$ given an initial approximation p_0:

INPUT initial approximation p_0; tolerance TOL; maximum number of iterations N_0.
OUTPUT approximate solution p or message of failure.
Step 1 Set $i = 1$.
Step 2 While $i \leq N_0$ do Steps 3–6.
  Step 3 Set $p = p_0 - f(p_0)/f'(p_0)$. (*Compute* p_i.)
  Step 4 If $|p - p_0| < TOL$ then
    OUTPUT (p); (*The procedure was successful.*)
    STOP.
  Step 5 Set $i = i + 1$.
  Step 6 Set $p_0 = p$. (*Update* p_0.)
Step 7 OUTPUT ('The method failed after N_0 iterations, $N_0 =$', N_0);
 (*The procedure was unsuccessful.*)
 STOP.

Newton's Illustration

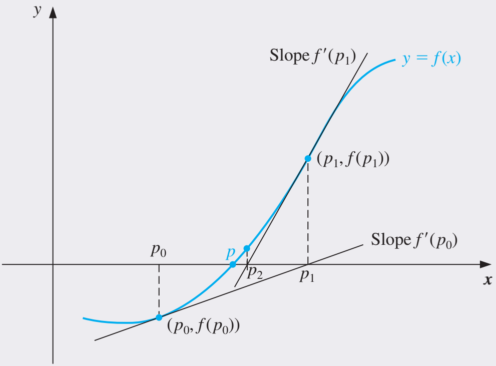

Figure: Figure 2.7

This YouTube video developed by MIT Open Courseware can serve as a good illustration of the Newton's Method for students. ▸ Newton's Method Video

Theorem (2.6)

Let $f \in C^2[a, b]$. If $p \in (a, b)$ is such that $f(p) = 0$ and $f'(p) \neq 0$, then there exists a $\delta > 0$ such that Newton's method generates a sequence $\{p_n\}_{n=1}^{\infty}$ converging to p for any initial approximation $p_0 \in [p - \delta, p + \delta]$.

Algorithm 2.4: SECANT METHOD

To find a solution to $f(x) = 0$ given initial approximations p_0 and p_1:

INPUT initial approximations p_0, p_1; tolerance TOL; maximum number of iterations N_0.
OUTPUT approximate solution p or message of failure.

Step 1 Set $i = 2$;
$q_0 = f(p_0)$;
$q_1 = f(p_1)$.

Step 2 While $i \leq N_0$ do Steps 3–6.

Step 3 Set $p = p_1 - q_1(p_1 - p_0)/(q_1 - q_0)$. (*Compute* p_i.)

Step 4 If $|p - p_1| < TOL$ then
OUTPUT (p); (*The procedure was successful.*)
STOP.

Step 5 Set $i = i + 1$.

Step 6 Set $p_0 = p_1$; (*Update* p_0, q_0, p_1, q_1.)
$q_0 = q_1$;
$p_1 = p$;
$q_1 = f(p)$.

Step 7 OUTPUT ('The method failed after N_0 iterations, $N_0 =$', N_0);
(*The procedure was unsuccessful.*)
STOP.

Secant Illustration

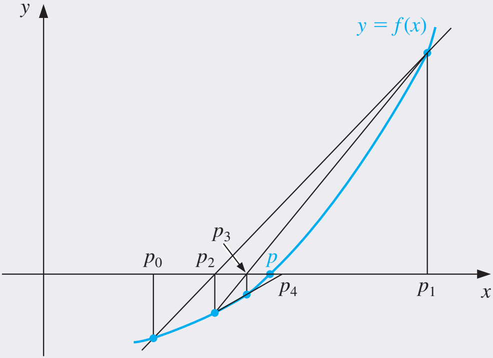

Figure: Figure 2.9

This YouTube video developed by Oscar Veliz can serve as a good illustration of the Secant Method for students.

Secant Method Video

Algorithm 2.5: FALSE POSITION

To find a solution to $f(x) = 0$ given the continuous function f on the interval $[p_0, p_1]$ where $f(p_0)$ and $f(p_1)$ have opposite signs:

INPUT initial approximations p_0, p_1; tolerance TOL; maximum number of iterations N_0.
OUTPUT approximate solution p or message of failure.
Step 1 Set $i = 2$;
$q_0 = f(p_0)$;
$q_1 = f(p_1)$.
Step 2 While $i \leq N_0$ do Steps 3–7.
Step 3 Set $p = p_1 - q_1(p_1 - p_0)/(q_1 - q_0)$. (*Compute p_i.*)
Step 4 If $|p - p_1| < TOL$ then
OUTPUT (p); (*The procedure was successful.*)
STOP.
Step 5 Set $i = i + 1$;
$q = f(p)$.
Step 6 If $q \cdot q_1 < 0$ then set $p_0 = p_1$;
$q_0 = q_1$.
Step 7 Set $p_1 = p$;
$q_1 = q$.
Step 8 OUTPUT ('Method failed after N_0 iterations, $N_0 =$', N_0);
(*The procedure unsuccessful.*)
STOP.

This YouTube video developed by Jacob Bishop can serve as a good illustration of the False Position Method for students.

False Position Method Video

Secant - Method of False Position Illustration

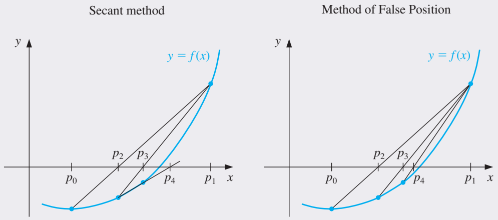

Figure: Figure 2.10

Definition (2.7: Order of Convergence)

Suppose $\{p_n\}_{n=0}^{\infty}$ is a sequence that converges to p, with $p_n \neq p$ for all n. If positive constants λ and α exist with

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^{\alpha}} = \lambda,$$

then $\{p_n\}_{n=0}^{\infty}$ **converges to p of order α, with asymptotic error constant λ.**

Theorem (2.8)

Let $g \in C[a,b]$ be such that $g(x) \in [a,b]$, for all $x \in [a,b]$. Suppose, in addition, that g' is continuous on (a,b) and a positive constant $k < 1$ exists with

$$|g'(x)| \leq k, \quad \text{for all } x \in (a,b).$$

If $g'(p) \neq 0$, then for any number $p_0 \neq p$ in $[a,b]$, the sequence

$$p_n = g(p_{n-1}), \quad \text{for } n \geq 1,$$

converges only linearly to the unique fixed point p in $[a,b]$.

Theorem (2.9)

Let p be a solution of the equation $x = g(x)$. Suppose that $g'(p) = 0$ and g'' is continuous with $|g''(x)| < M$ on an open interval I containing p. Then there exists a $\delta > 0$ such that, for $p_0 \in [p - \delta, p + \delta]$, the sequence defined by $p_n = g(p_{n-1})$, when $n \geq 1$, converges at least quadratically to p. Moreover, for sufficiently large values of n,

$$|p_{n+1} - p| < \frac{M}{2}|p_n - p|^2.$$

Definition (2.10: Multiplicity)

A solution p of $f(x) = 0$ is a **zero of multiplicity** m of f if for $x \neq p$, we can write $f(x) = (x - p)^m q(x)$, where $\lim_{x \to p} q(x) \neq 0$.

Theorem (2.11)

The function $f \in C^1[a, b]$ has a simple zero at p in (a, b) if and only if $f(p) = 0$, but $f'(p) \neq 0$.

Theorem (2.12)

The function $f \in C^m[a, b]$ has a zero of multiplicity m at p in (a, b) if and only if

$$0 = f(p) = f'(p) = f''(p) = \cdots = f^{(m-1)}(p), \quad \text{but} \quad f^{(m)}(p) \neq 0.$$

Aitken's Δ^2 Method

Aitken's Δ^2 method is based on the assumption that the sequence $\{\hat{p}_n\}_{n=0}^{\infty}$, defined by

$$\hat{p}_n = p_n - \frac{(p_{n+1} - p_n)^2}{p_{n+2} - 2p_{n+1} + p_n},$$

converges more rapidly to p than does the original sequence $\{p_n\}_{n=0}^{\infty}$.

Definition (2.13)

For a given sequence $\{p_n\}_{n=0}^{\infty}$, the **forward difference** Δp_n (read "delta p_n") is defined by

$$\Delta p_n = p_{n+1} - p_n, \quad \text{for } n \geq 0.$$

Higher powers of the operator Δ are defined recursively by

$$\Delta^k p_n = \Delta(\Delta^{k-1} p_n), \quad \text{for } k \geq 2.$$

Theorem (2.14)

Suppose that $\{p_n\}_{n=0}^{\infty}$ is a sequence that converges linearly to the limit p and that

$$\lim_{n\to\infty} \frac{p_{n+1} - p}{p_n - p} < 1.$$

Then the Aitken's Δ^2 sequence $\{\hat{p}_n\}_{n=0}^{\infty}$ converges to p faster than $\{p_n\}_{n=0}^{\infty}$ in the sense that

$$\lim_{n\to\infty} \frac{\hat{p}_n - p}{p_n - p} = 0.$$

Algorithm 2.6: STEFFENSEN'S METHOD

To find a solution to $p = g(p)$ given an initial approximation p_0:

INPUT initial approximation p_0; tolerance TOL; maximum number of iterations N_0.
OUTPUT approximate solution p or message of failure.
Step 1 Set $i = 1$.
Step 2 While $i \leq N_0$ do Steps 3–6.

Step 3 Set $p_1 = g(p_0)$; (*Compute* $p_1^{(i-1)}$.)
$p_2 = g(p_1)$; (*Compute* $p_2^{(i-1)}$.)
$p = p_0 - (p_1 - p_0)^2/(p_2 - 2p_1 + p_0)$. (*Compute* $p_0^{(i)}$.)

Step 4 If $|p - p_0| < TOL$ then
OUTPUT (p); (*Procedure completed successfully*.)
STOP.

Step 5 Set $i = i + 1$.

Step 6 Set $p_0 = p$. (*Update* p_0.)

Step 7 OUTPUT ('Method failed after N_0 iterations, $N_0 =$', N_0);
(*Procedure completed unsuccessfully*.)
STOP.

k	$p_0^{(k)}$	$p_1^{(k)}$	$p_2^{(k)}$
0	$p_0^{(0)}$	$p_1^{(0)} = g(p_0^{(0)})$	$p_2^{(0)} = g(p_1^{(0)})$
1	$p_0^{(1)} = p_0^{(0)} - \dfrac{(p_1^{(0)} - p_0^{(0)})^2}{p_2^{(0)} - 2p_1^{(0)} + p_0^{(0)}}$	$p_1^{(1)} = g(p_0^{(1)})$	$p_2^{(1)} = g(p_1^{(1)})$
2	$p_0^{(2)} = p_0^{(1)} - \dfrac{(p_1^{(1)} - p_0^{(1)})^2}{p_2^{(1)} - 2p_2^{(1)} + p_0^{(1)}}$		

Table: Table 2.11

Theorem (2.15)

Suppose that $x = g(x)$ *has the solution* p *with* $g'(p) \neq 1$*. If there exists a* $\delta > 0$ *such that* $g \in C^3[p-\delta, p+\delta]$*, then Steffensen's method gives quadratic convergence for any* $p_0 \in [p-\delta, p+\delta]$*.*

Theorem (2.16: Fundamental Theorem of Algebra)

If $P(x)$ is a polynomial of degree $n \geq 1$ with real or complex coefficients, then $P(x) = 0$ has at least one (possibly complex) root.

Corollary (2.17)

If $P(x)$ is a polynomial of degree $n \geq 1$ with real or complex coefficients, then there exist unique constants $x_1, x_2, \ldots, x_k$, possibly complex, and unique positive integers $m_1, m_2, \ldots, m_k$, such that $\sum_{i=1}^{k} m_i = n$ and

$$P(x) = a_n(x - x_1)^{m_1}(x - x_2)^{m_2} \cdots (x - x_k)^{m_k}.$$

Corollary (2.18)

Let $P(x)$ and $Q(x)$ be polynomials of degree at most n. If x_1, $x_2, \ldots, x_k$, with $k > n$, are distinct numbers with $P(x_i) = Q(x_i)$ for $i = 1, 2, \ldots, k$, then $P(x) = Q(x)$ for all values of x.

Theorem (2.19: Horner's Method)

Let

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0.$$

Define $b_n = a_n$ *and*

$$b_k = a_k + b_{k+1} x_0, \quad \text{for } k = n-1, n-2, \ldots, 1, 0.$$

Then $b_0 = P(x_0)$. *Moreover, if*

$$Q(x) = b_n x^{n-1} + b_{n-1} x^{n-2} + \cdots + b_2 x + b_1,$$

then

$$P(x) = (x - x_0) Q(x) + b_0.$$

Algorithm 2.7: HORNER'S METHOD

To evaluate the polynomial

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = (x - x_0)Q(x) + b_0$$

and its derivative at x_0:

INPUT degree n; coefficients $a_0, a_1, \ldots, a_n$; x_0.
OUTPUT $y = P(x_0)$; $z = P'(x_0)$.
Step 1 Set $y = a_n$; ($\textit{Compute } b_n \textit{ for } P.$)
$z = a_n$. ($\textit{Compute } b_{n-1} \textit{ for } Q.$)
Step 2 For $j = n-1, n-2, \ldots, 1$
set $y = x_0 y + a_j$; ($\textit{Compute } b_j \textit{ for } P.$)
$z = x_0 z + y$. ($\textit{Compute } b_{j-1} \textit{ for } Q.$)
Step 3 Set $y = x_0 y + a_0$. ($\textit{Compute } b_0 \textit{ for } P.$)
Step 4 OUTPUT (y, z);
STOP.

This YouTube video developed by We Teach Academy can serve as a good illustration of the Horner's Method for students.

False Position Method Video

Theorem (2.20)

If $z = a + bi$ is a complex zero of multiplicity m of the polynomial $P(x)$ with real coefficients, then $\overline{z} = a - bi$ is also a zero of multiplicity m of the polynomial $P(x)$, and $(x^2 - 2ax + a^2 + b^2)^m$ is a factor of $P(x)$.

Algorithm 2.8: MÜLLER'S METHOD

To find a solution to $f(x) = 0$ given three approximations, p_0, p_1, and p_2:

INPUT p_0, p_1, p_2; tolerance TOL; maximum number of iterations N_0.
OUTPUT approximate solution p or message of failure.

Step 1 Set $h_1 = p_1 - p_0$; $h_2 = p_2 - p_1$; $\delta_1 = (f(p_1) - f(p_0))/h_1$;
$\delta_2 = (f(p_2) - f(p_1))/h_2$; $d = (\delta_2 - \delta_1)/(h_2 + h_1)$; $i = 3$.

Step 2 While $i \leq N_0$ do Steps 3–7.

Step 3 $b = \delta_2 + h_2 d$; $D = (b^2 - 4f(p_2)d)^{1/2}$. (*May require complex arithmetic.*)

Step 4 If $|b - D| < |b + D|$ then set $E = b + D$
else set $E = b - D$.

Step 5 Set $h = -2f(p_2)/E$; $p = p_2 + h$.

Step 6 If $|h| < TOL$ then
OUTPUT (p); (*The procedure was successful.*)
STOP.

Step 7 Set $p_0 = p_1$; (*Prepare for next iteration.*)
$p_1 = p_2$; $p_2 = p$; $h_1 = p_1 - p_0$; $h_2 = p_2 - p_1$; $\delta_1 = (f(p_1) - f(p_0))/h_1$;
$\delta_2 = (f(p_2) - f(p_1))/h_2$; $d = (\delta_2 - \delta_1)/(h_2 + h_1)$; $i = i + 1$.

Step 8 OUTPUT ('Method failed after N_0 iterations, $N_0 =$', N_0);
(*The procedure was unsuccessful.*)
STOP.

This YouTube video developed by Jacob Bishop can serve as a good illustration of the Müller's Method for students.

▸ Müller's Method Video

This website from the University of Waterloo provides students with two numerical examples of Müller's Method .

▸ Müller's Method Examples