Math 541 - Numerical Analysis

Lecture Notes – Calculus and Taylor's Theorem

Joseph M. Mahaffy,
⟨jmahaffy@mail.sdsu.edu⟩

Department of Mathematics and Statistics
Dynamical Systems Group
Computational Sciences Research Center
San Diego State University
San Diego, CA 92182-7720

http://jmahaffy.sdsu.edu

Spring 2018

Outline

Why Review Calculus???

It's a good warm-up for our brains!

When developing numerical schemes we will use theorems from calculus to guarantee that our algorithms make sense.

If the theory is sound, when our programs fail we look for bugs in the code!

Background Material — A Crash Course in Calculus

Key concepts from Calculus

- Limits
- Continuity
- Differentiability
- **Taylor's Theorem**

Limit

The most fundamental concept in Calculus is the **limit**.

Definition (Limit)

A function f defined on a set X of real numbers $X \subset \mathbb{R}$ has the limit L at x_0, written

$$\lim_{x \to x_0} f(x) = L$$

if given any real number $\epsilon > 0$, there exists a real number $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $x \in X$ and $0 < |x - x_0| < \delta$.

Continuity

Definition (Continuity (at a point))

Let f be a function defined on a set X of real numbers, and $x_0 \in X$. Then f is continuous at x_0 if

$$\lim_{x \to x_0} f(x) = f(x_0).$$

It is important to note that computers only have **discrete** representation, not **continuous**.

Thus, the computer is often making approximations.

Derivative

Definition (Differentiability (at a point))

Let f be a function defined on an open interval containing x_0 $(a < x_0 < b)$. f is differentiable at x_0 if

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \quad \text{exists.}$$

If the limit exists, $f'(x_0)$ is the derivative at x_0. **Note:** This is the **slope** of the tangent line at $f(x_0)$.

The derivative is used often in this course, and sometimes an approximate derivative is adequate.

Taylor's Theorem

The following theorem is the most important one for you to remember from Calculus.

Theorem (Taylor's Theorem with Remainder)

Suppose $f \in \mathcal{C}^n[a,b]$, $f^{(n+1)}$ *exists on* $[a,b]$, *and* $x_0 \in [a,b]$. *Then for all* $x \in (a,b)$, *there exists* $\xi(x) \in (x_0, x)$ *with* $f(x) = P_n(x) + R_n(x)$ *where*

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k,$$

$$R_n(x) = \frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x - x_0)^{(n+1)}.$$

$P_n(x)$ *is called the* ***Taylor polynomial of degree*** n, *and* $R_n(x)$ *is the* ***remainder term*** *(truncation error).*

Note: $f^{(n+1)}$ exists on $[a,b]$, but is not necessarily continuous.

Important Examples

Important Examples: Below are important functions studied in Calculus

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

Example 1: Approximate sin

Example 1: Approximate $\sin(x)$ with x near $\frac{\pi}{6}$

We know $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$, so what about $\sin\left(\frac{\pi}{6} + 0.1\right)$

Since $f(x) \in \mathcal{C}^{\infty}(-\infty, \infty)$, we can use Taylor's theorem:

$$\begin{aligned} f(x) &= f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2!}f''(x_0)(x - x_0)^2 + \ldots \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(x_0)(x - x_0)^n \end{aligned}$$

Example 1: Approximate sin 2

From Taylor's theorem $\sin(x)$ with x near $\frac{\pi}{6}$

$$
\begin{aligned}
\sin(x) &= \sum_{n=0}^{\infty} \frac{1}{n!} \frac{d^n}{dx^n} \sin(x)\bigg|_{x=\frac{\pi}{6}} \left(x - \frac{\pi}{6}\right)^n \\
&= \\
&= \sin\left(\frac{\pi}{6}\right) + \cos\left(\frac{\pi}{6}\right)\left(x - \frac{\pi}{6}\right) - \\
&\quad \frac{1}{2!}\sin\left(\frac{\pi}{6}\right)\left(x - \frac{\pi}{6}\right)^2 - \frac{1}{3!}\cos\left(\frac{\pi}{6}\right)\left(x - \frac{\pi}{6}\right)^3 + \ldots
\end{aligned}
$$

But $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$ and $\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$

Example 1: Approximate sin

With information above and $x = \frac{\pi}{6} + 0.1$, we have

$$\begin{aligned}\sin(x) &= \frac{1}{2}\left[1 - \frac{1}{2!}\left(x - \frac{\pi}{6}\right)^2 + \frac{1}{4!}\left(x - \frac{\pi}{6}\right)^4 - \dots\right] \\ &\quad + \frac{\sqrt{3}}{2}\left[\left(x - \frac{\pi}{6}\right) - \frac{1}{3!}\left(x - \frac{\pi}{6}\right)^3 + \dots\right] \\ &= \frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\left(x - \frac{\pi}{6}\right)^{2n} + \frac{\sqrt{3}}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\left(x - \frac{\pi}{6}\right)^{2n+1}\end{aligned}$$

It follows that $\sin\left(\frac{\pi}{6} + 0.1\right)$ satisfies:

$$\sin\left(\frac{\pi}{6} + 0.1\right) = \frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}(0.1)^{2n} + \frac{\sqrt{3}}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}(0.1)^{2n+1}$$

Example 1: Approximate sin

Examining the infinite sums, we see both the $(0.1)^n$ and the factorials in the denominator resulting terms going to **zero**

We truncate the series at $n = N$, gives the approximation at $x = \frac{\pi}{6} + 0.1$

$$\sin\left(\frac{\pi}{6} + 0.1\right) \approx \frac{1}{2}\sum_{n=0}^{N}\frac{(-1)^n}{(2n)!}(0.1)^{2n} + \frac{\sqrt{3}}{2}\sum_{n=0}^{N}\frac{(-1)^n}{(2n+1)!}(0.1)^{2n+1}$$

Truncating the series at $n = N$ leaves a polynomial of order $2N + 1$

$$T_N(x) = \frac{1}{2}\sum_{n=0}^{N}\frac{(-1)^n}{(2n)!}\left(x - \frac{\pi}{6}\right)^{2n} + \frac{\sqrt{3}}{2}\sum_{n=0}^{N}\frac{(-1)^n}{(2n+1)!}\left(x - \frac{\pi}{6}\right)^{2n+1},$$

where x is "close" to $\frac{\pi}{6}$

Example 1: Approximate sin

The error or remainder satisfies:

$$R_N(x) = \frac{1}{2}\sum_{n=N+1}^{\infty}\frac{(-1)^n}{(2n)!}\left(x-\frac{\pi}{6}\right)^{2n} + \frac{\sqrt{3}}{2}\sum_{n=N+1}^{\infty}\frac{(-1)^n}{(2n+1)!}\left(x-\frac{\pi}{6}\right)^{2n+1},$$

Thus, $\sin\left(\frac{\pi}{6}+0.1\right) = T_N\left(\frac{\pi}{6}+0.1\right) + R_N\left(\frac{\pi}{6}+0.1\right)$

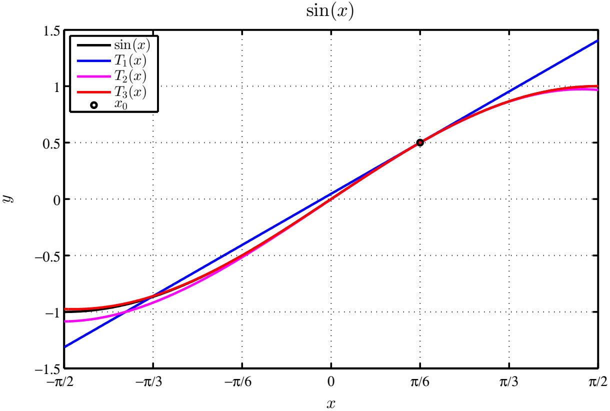
If we use the approximation from the previous page with $x = \frac{\pi}{6}+0.1$, we find the following polynomial evaluations:

	Poly Order	Approximation	Error
$\sin\left(\frac{\pi}{6}+0.1\right)$	∞	0.58396036	
$T_1\left(\frac{\pi}{6}+0.1\right)$	1	0.58660254	0.45246%
$T_2\left(\frac{\pi}{6}+0.1\right)$	3	0.58395820	-0.00037%
$T_3\left(\frac{\pi}{6}+0.1\right)$	5	0.58396036	8.56×10^{-8}%

SDSU

Example 1: Approximate sin

Below is the graph of $y = \sin(x)$ with the Taylor polynomial fits of order 1, 3, and 5, passing through $x_0 = \frac{\pi}{6}$



We observe even a **cubic polynomial** fits the sine function well

Example 1: Approximate sin

The remainder term in **Taylor's theorem** is useful for finding bounds on the error.

Recall

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)^{(n+1)}$$

with $\xi \in (x_0, x)$. However, we rarely know ξ.

A bound on the error satisfies

$$\begin{aligned} \max_{x \in [x_0-\delta, x_0+\delta]} |R_n(x)| &= \max_{x \in [x_0-\delta, x_0+\delta]} \frac{|f^{(n+1)}(\xi)|}{(n+1)!} |x - x_0|^{(n+1)} \\ &\leq \frac{\delta^{n+1}}{(n+1)!} \max_{x \in [x_0-\delta, x_0+\delta]} |f^{(n+1)}(\xi)| \end{aligned}$$

Example 1: Approximate sin

For this example, the $(n+1)^{st}$ derivative of $f(x) = \sin(x)$ satisfies

$$|f^{(n+1)}(\xi)| \le 1,$$

and we are taking $\delta = 0.1$

It follows that

$$\max_{x\in[x_0-\delta,x_0+\delta]} |R_n(x)| \le \frac{\delta^{n+1}}{(n+1)!} \max_{x\in[x_0-\delta,x_0+\delta]} |f^{(n+1)}(\xi)| \le \frac{\delta^{n+1}}{(n+1)!} \le \frac{(0.1)^{n+1}}{(n+1)!}$$

We saw the error for $T_2(x)$ (cubic fit) was 2.16×10^{-6}.

The error approximation gives

$$E_3(x) \le \frac{(0.1)^4}{4!} \approx 4.17 \times 10^{-6},$$

which is only double the actual error

Example 2: Integrate $\cos(\cos(x))$ 1

Example 2: Consider the following integral:

$$\int_0^{\frac{\pi}{2}} \cos(\cos(x))dx$$

- This is not an integral that is readily solvable with standard methods
- Can we obtain a reasonable approximation?
- **Maple** and **MatLab** can numerically solve this problem
- Later in the course we learn quadrature methods for solving
- **Polynomials** are easy to integrate, so let's try using **Taylor's theorem** and integrate the truncated polynomial.

Example 2: Integrate $\cos(\cos(x))$ 2

Our function is clearly $\mathcal{C}^\infty(-\infty, \infty)$, so **Taylor's theorem** readily applies

$$f(x) = \cos(\cos(x)) = \sum_0^\infty \frac{1}{n!}\frac{d^n f(0)}{dx^n}x^n$$

There is no easy form for $\frac{d^n f(0)}{dx^n}$, but taking a few terms is not hard

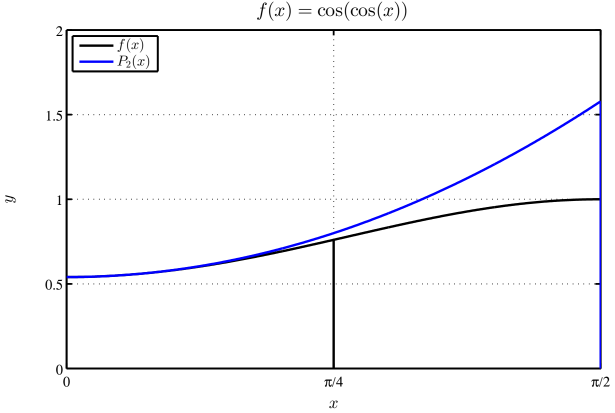
$$\begin{aligned} f(0) &= \cos(\cos(0)) = \cos(1) \\ f'(0) &= \sin(\cos(0))\sin(0) = 0 \\ f''(0) &= -\cos(\cos(0))\sin^2(0) + \sin(\cos(0))\cos(0) = \sin(1) \end{aligned}$$

It follows that a **quadratic approximating polynomial** is:

$$f(x) \approx P_2(x) = \cos(1) + \frac{\sin(1)}{2}x^2$$

Example 2: Integrate $\cos(\cos(x))$

The integral gives the area under the curve. The figure below shows $f(x) = \cos(\cos(x))$ and the second order Maclaurin series expansion $P_2(x) = \cos(1) + \frac{\sin(1)}{2}x^2$



Example 2: Integrate $\cos(\cos(x))$

The 2^{nd} order Maclaurin series expansion $P_2(x) = \cos(1) + \frac{\sin(1)}{2}x^2$ is easily integrable

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} \cos(\cos(x))dx &\approx \int_0^{\frac{\pi}{2}} \left(\cos(1) + \frac{\sin(1)}{2}x^2\right) dx \\
&= \left(\cos(1)x + \frac{\sin(1)x^3}{6}\right)\bigg|_0^{\frac{\pi}{2}} \\
&= \frac{\pi\cos(1)}{2} + \frac{\pi^3\sin(1)}{48} \approx 1.392265,
\end{aligned}$$

which is larger than the actual value (1.201970) as seen in the graph. This is a 15.8% error, so not great.

Example 2: Integrate $\cos(\cos(x))$

How much is the error improved if the interval is divided into two equal intervals?

This time we use Taylor's expansions around $x_0 = 0$ and $x_0 = \frac{\pi}{4}$, and again truncate with 2^{nd} order polynomials

About $x_0 = \frac{\pi}{4}$, Taylor's series is

$$\begin{aligned} T_2(x) &= \cos\left(\frac{\sqrt{2}}{2}\right) + \frac{\sqrt{2}\sin\left(\frac{\sqrt{2}}{2}\right)}{2}\left(x - \frac{\pi}{4}\right) \\ &\quad + \left(\frac{\sqrt{2}\sin\left(\frac{\sqrt{2}}{2}\right)}{4} - \frac{\cos\left(\frac{\sqrt{2}}{2}\right)}{4}\right)\left(x - \frac{\pi}{4}\right)^2 \\ &\approx 0.76024 + 0.45936\left(x - \frac{\pi}{4}\right) + 0.039620\left(x - \frac{\pi}{4}\right)^2 \end{aligned}$$

Example 2: Integrate $\cos(\cos(x))$

The integral is now approximated by

$$\int_0^{\frac{\pi}{4}} \cos(\cos(x))dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos(\cos(x))dx \approx \int_0^{\frac{\pi}{4}} P_2(x)dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} T_2(x)dx,$$

where

$$\begin{aligned} P_2(x) &\approx 0.54030 + 0.42074\,x^2 \quad \text{and} \\ T_2(x) &\approx 0.76024 + 0.45936\left(x - \frac{\pi}{4}\right) + 0.039620\left(x - \frac{\pi}{4}\right)^2 \end{aligned}$$

However, integrating these quadratic polynomials is easy

$$\int_0^{\frac{\pi}{4}} P_2(x)dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} T_2(x)dx \approx 0.492297 + 0.745172 = 1.237469,$$

which is only a 2.95% error from the actual value

Example 2: Integrate $\cos(\cos(x))$

The figure below shows a diagram for the computations done above with two **approximating quadratics** for finding the area