Math 541 - Numerical Analysis

Interpolation and Polynomial Approximation — Piecewise Polynomial Approximation; Cubic Splines

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Outline

An n-degree polynomial passing through $n+1$ points

Polynomial Interpolation

Construct a polynomial passing through the points $(x_0, f(x_0))$, $(x_1, f(x_1))$, $(x_2, f(x_2))$, $\ldots$, $(x_N, f(x_n))$.

Define $L_{n,k}(x)$, the **Lagrange coefficients:**

$$\mathbf{L_{n,k}(x)} = \prod_{\mathbf{i=0,\, i\neq k}}^{\mathbf{n}} \frac{\mathbf{x - x_i}}{\mathbf{x_k - x_i}} = \frac{x - x_0}{x_k - x_0} \cdots \frac{x - x_{k-1}}{x_k - x_{k-1}} \cdot \frac{x - x_{k+1}}{x_k - x_{k+1}} \cdots \frac{x - x_n}{x_k - x_n},$$

which have the properties

$$L_{n,k}(x_k) = 1; \quad L_{n,k}(x_i) = 0, \quad \text{for all } i \neq k.$$

The n^{th} Lagrange Interpolating Polynomial

We use $L_{n,k}(x)$, $k = 0, \ldots, n$ as building blocks for the ***Lagrange interpolating polynomial***:

$$P(x) = \sum_{k=0}^{n} f(x_k) L_{n,k}(x),$$

which has the property

$$P(x_i) = f(x_i), \quad \text{for all } i = 0, \ldots, n.$$

This is the unique n^{th} degree polynomial passing through the points
$(x_i, f(x_i))$, $i = 0, \ldots, n$**.**

Combining Taylor and Lagrange Polynomials

A ***Taylor polynomial of degree*** n matches the function and its first n derivatives at one point.

A ***Lagrange polynomial of degree*** n matches the function values at $n+1$ points.

Question: Can we combine the ideas of Taylor and Lagrange to get an interpolating polynomial that matches both the function values and some number of derivatives at multiple points?

Answer: To our euphoric joy, such polynomials exist! They are called ***Osculating Polynomials.***

The Concise Oxford Dictionary:

Osculate 1. (arch. or joc.) kiss. **2.** (Biol., of species, etc.) be related through intermediate species etc., have common characteristics *with* another or with each other. **3.** (Math., of curve or surface) have contact of higher than first order with, meet at three or more coincident points.

Osculating Polynomials

In Painful Generality

Given $(n+1)$ distinct points $\{x_0, x_1, \ldots, x_n\} \in [a, b]$, and non-negative integers $\{m_0, m_1, \ldots, m_n\}$.

Notation: Let $m = \max\{m_0, m_1, \ldots, m_n\}$.

The ***osculating polynomial approximation*** of a function $f \in C^m[a, b]$ at $x_i,\ i = 0, 1, \ldots, n$ is the polynomial (of lowest possible order) that agrees with

$$\{f(x_i), f'(x_i), \ldots, f^{(m_i)}(x_i)\} \text{ at } x_i \in [a, b],\ \forall i.$$

The degree of the osculating polynomial is **at most**

$$M = n + \sum_{i=0}^{n} m_i.$$

In the case where $m_i = 1,\ \forall i$ the polynomial is called a **Hermite Interpolatory Polynomial**.

Hermite Interpolatory Polynomials — The Existence Statement

If $f \in C^1[a,b]$ and $\{x_0, x_1, \ldots, x_n\} \in [a,b]$ are distinct, the unique polynomial of least degree ($\leq 2n+1$) agreeing with $f(x)$ and $f'(x)$ at $\{x_0, x_1, \ldots, x_n\}$ is

$$\mathbf{H_{2n+1}(x) = \sum_{j=0}^{n} f(x_j) H_{n,j}(x) + \sum_{j=0}^{n} f'(x_j) \hat{H}_{n,j}(x)},$$

where

$$H_{n,j}(x) = \left[1 - 2(x - x_j) L'_{n,j}(x_j)\right] L^2_{n,j}(x)$$

$$\hat{H}_{n,j}(x) = (x - x_j) L^2_{n,j}(x),$$

and $L_{n,j}(x)$ are our old friends, the ***Lagrange coefficients***:

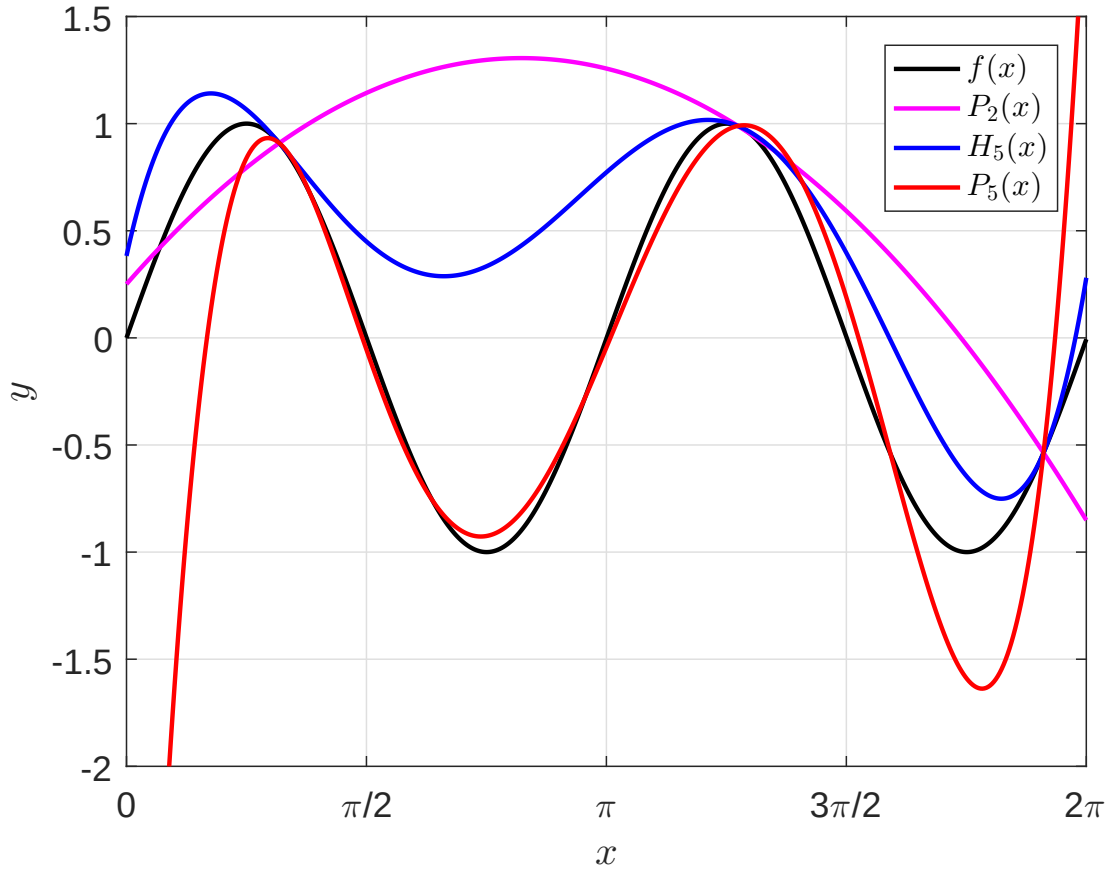
$$L_{n,j}(x) = \prod_{i=0,\, i \neq j}^{n} \frac{x - x_i}{x_j - x_i}.$$

Further, if $f \in C^{2n+2}[a,b]$, then for some $\xi(x) \in [a,b]$

$$f(x) = H_{2n+1}(x) + \frac{\prod_{i=0}^{n} (x - x_i)^2}{(2n+2)!} f^{(2n+2)}(\xi(x)).$$

Example of Interpreting Polynomials

Consider the function $f(x) = \sin(2x)$. The graph below shows fits of 3 and 6 points with Lagrange $P_2(x)$ and $P_5(x)$, respectively, and 3 points, x_i, with $f(x_i)$ and $f'(x_i)$, using Hermite $H_5(x)$ polynomials.



Checking the Roadmap

Inspired by Weierstrass, we have looked at a number of strategies for approximating arbitrary functions using polynomials.

Taylor	Detailed information from one point, excellent locally, but not very successful for extended intervals.
Lagrange	$\leq n$th degree poly. interpolating the function in $(n+1)$ pts. ***Representation:*** Theoretical using the Lagrange coefficients $L_{n,k}(x)$
Hermite	$\leq (2n+1)$th degree polynomial interpolating the function, and matching its first derivative in $(n+1)$ points. ***Representation:*** Theoretical using two types of Hermite coefficients $H_{n,k}(x)$, and $\widehat{H}_{n,k}(x)$

With $(n+1)$ points, and a uniform matching criteria of m derivatives in each point we can talk about these in terms of the broader class of ***osculating polynomials*** with:

Taylor(m,n=0), Lagrange(m=0,n), Hermite(m=1,n); with resulting degree $d \leq (m+1)(n+1)-1$.

Admiring the Roadmap... Are We Done?

There are many methods (Neville, Newton's divided difference) to produce representations of arbitrary osculating polynomials...

We have swept a dirty little secret under the rug: —

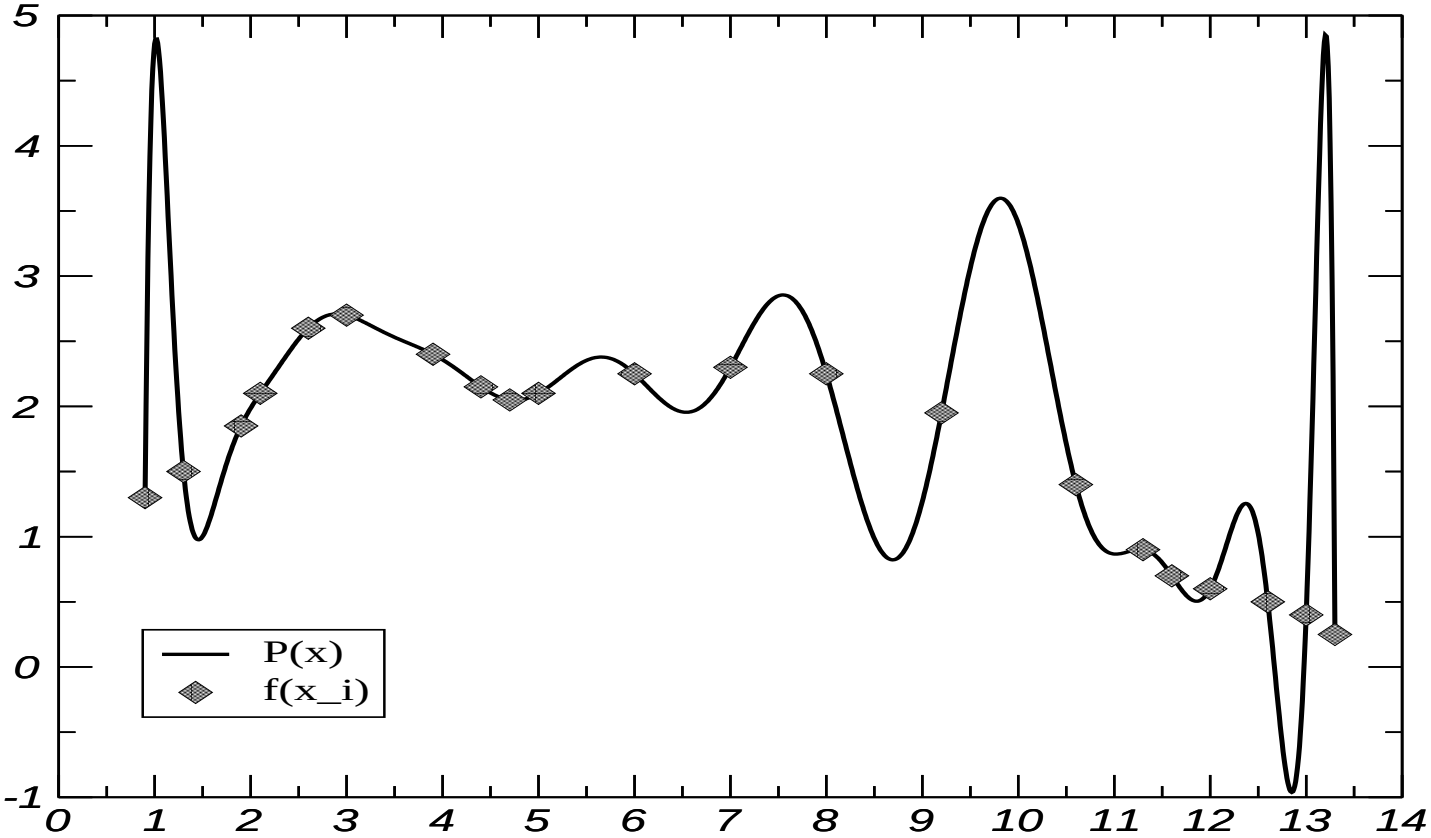
For all these interpolation strategies we get — provided the underlying function is smooth enough, *i.e.* $f \in C^{(m+1)(n+1)}([a,b])$ — errors of the form

$$\underbrace{\frac{\prod_{i=0}^{n}(x-x_i)^{(m+1)}}{((m+1)(n+1))!}}_{\eta(x)} f^{((m+1)(n+1))}(\xi(x)), \quad \xi(x) \in [a,b]$$

We have seen that with the x_i's dispersed (Lagrange / Hermite-style), the controllable part, $\eta(x)$, of the error term is better behaved than for Taylor polynomials (but is it well-behaved enough?!) ***However,*** we have **no** control over the $((n+1)(m+1))$th derivative of f.

Problems with High Order Polynomial Approximation

We can force a polynomial of high degree to pass through as many points $(x_i, f(x_i))$ as we like. However, high degree polynomials tend to fluctuate wildly ***between*** the interpolating points.



Alternative Approach to Interpolation

The oscillations tend to be extremely bad close to the **end points** of the interval of interest, and (in general) the more points you put in, the wilder the oscillations get!

Clearly, we need some new tricks!

Idea: Divide the interval into smaller sub-intervals, and construct different low degree polynomial approximations (with small oscillations) on the sub-intervals.

This is called ***Piecewise Polynomial Approximation***.

Simplest continuous variant: ***Piecewise Linear Approximation***:

Piecewise Linear Approximation

Connect-the-Dots

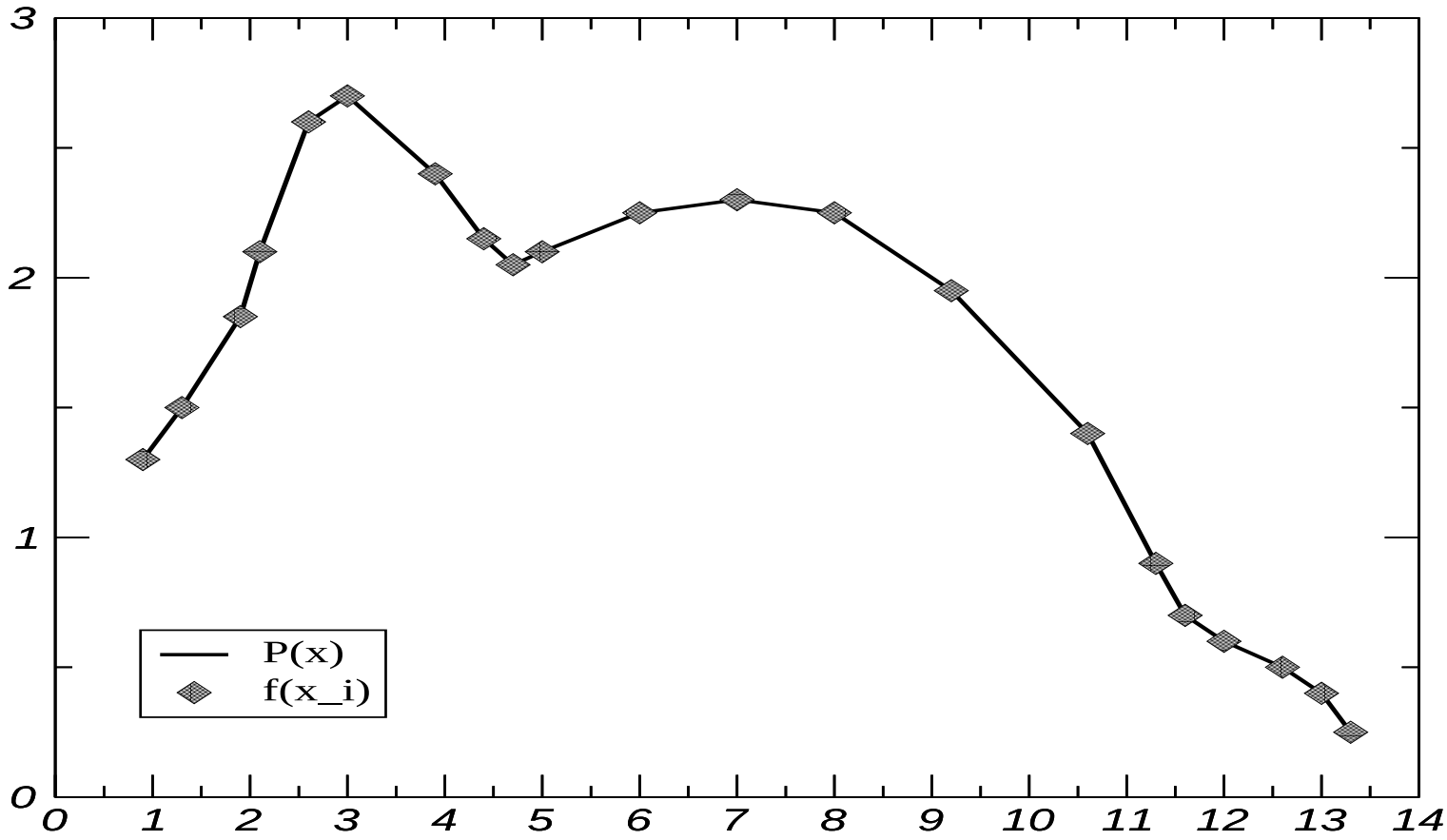


Figure: Piecewise linear approximation of the same data as on slide 11. Is this the end of excessive oscillations?!?

Problem with Piecewise Linear Approximation

The piecewise linear interpolating function is ***not differentiable*** at the ***"nodes,"*** *i.e.* the points x_i. (Typically we want to do more than just plot the polynomial... and even plotting shows sharp corners!)

Idea: Strengthened by our experience with Hermite polynomials, why not generate piecewise polynomials that match both the function value and some number of derivatives in the nodes!

The Return of the Cubic Hermite Polynomial!

If, for instance $f(x)$ and $f'(x)$ are known in the nodes, we can use a collection of ***cubic Hermite polynomials*** $H_j^3(x)$ to build up such a function.

But... what if $f'(x)$ is ***not*** known (in general getting measurements of the derivative of a physical process is much more difficult and unreliable than measuring the quantity itself), can we still generate an interpolant with continuous derivative(s)???

An Old Idea: Splines

(Edited for Space, and "Content") Wikipedia Definition: Spline —

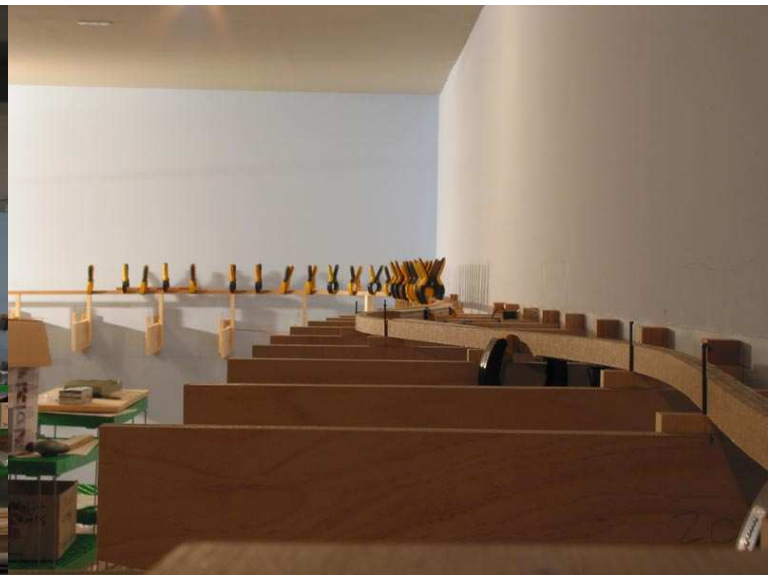
A spline consists of a long strip of wood (a lath) fixed in position at a number of points. Shipwrights often used splines to mark the curve of a hull. The lath will take ***the shape which minimizes the energy required for bending it between the fixed points***, and thus adopt the smoothest possible shape.

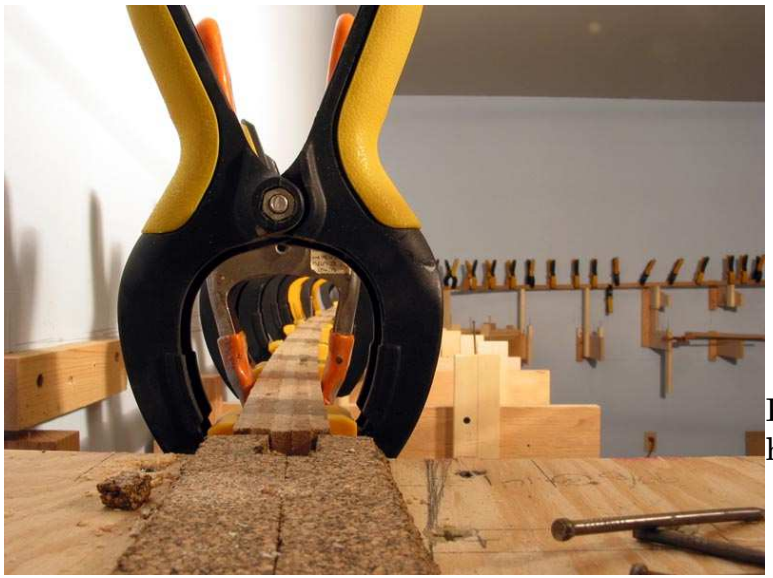
Later craftsmen have made splines out of rubber, steel, and other elastomeric materials.

Spline devices help bend the wood for pianos, violins, violas, etc. The Wright brothers used one to shape the wings of their aircraft.

In 1946 mathematicians started studying the spline shape, and derived the ***piecewise polynomial formula known as the spline curve or function.*** This has led to the widespread use of such functions in ***computer-aided design, especially in the surface designs of vehicles.***

Modern Spline Construction: — A Model Railroad





Pictures from Charlie Comstock's webpage
http://s145079212.onlinehome.us/rr/

Applications & Pretty Pictures

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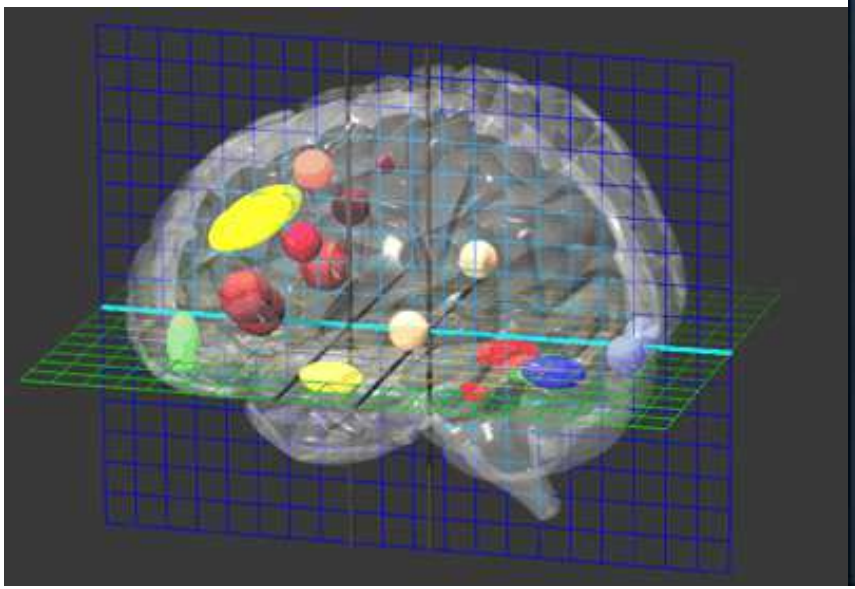

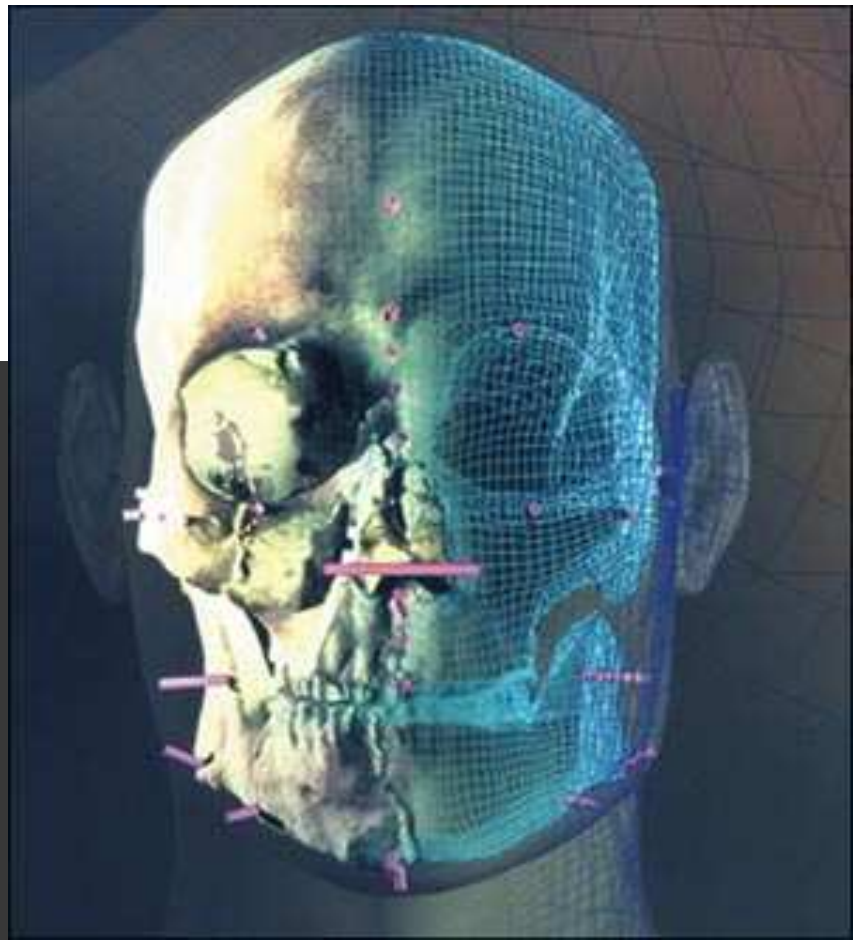

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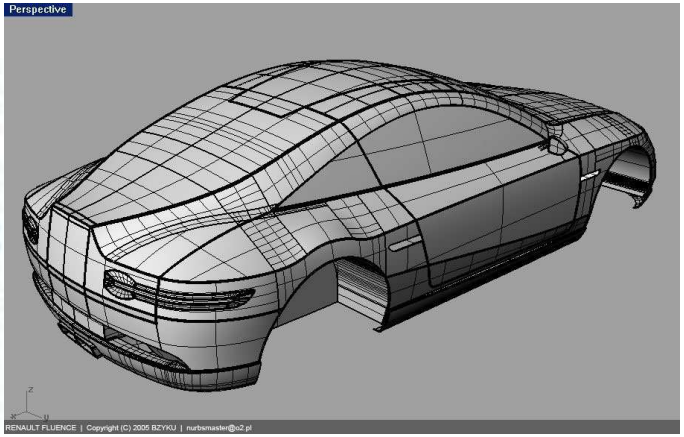
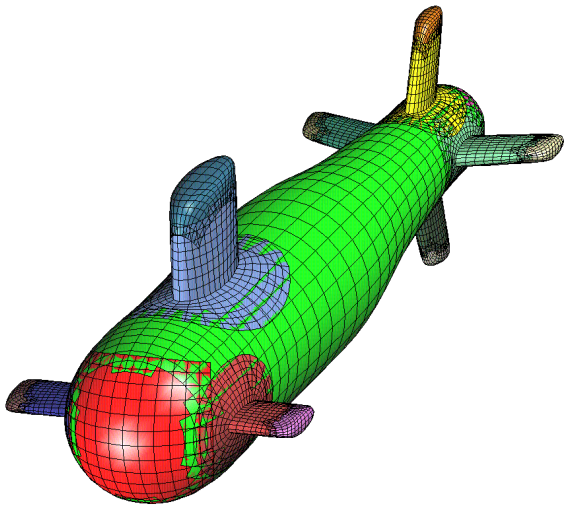
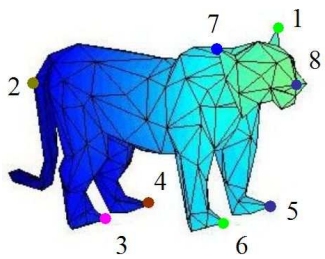
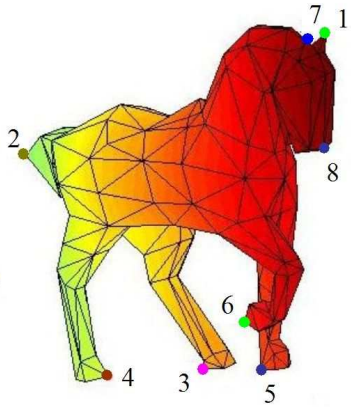








Cubic Splines to the Rescue!!! 1D-version

Given a function f defined on $[a, b]$ and a set of nodes $a = x_0 < x_1 < \cdots < x_n = b$, a ***cubic spline interpolant*** *S for f is a function that satisfies the following conditions:*

a. $S(x)$ *is a cubic polynomial, denoted* $S_j(x)$, *on the sub-interval* $[x_j, x_{j+1}]$ $\forall j = 0, 1, \ldots, n-1$.

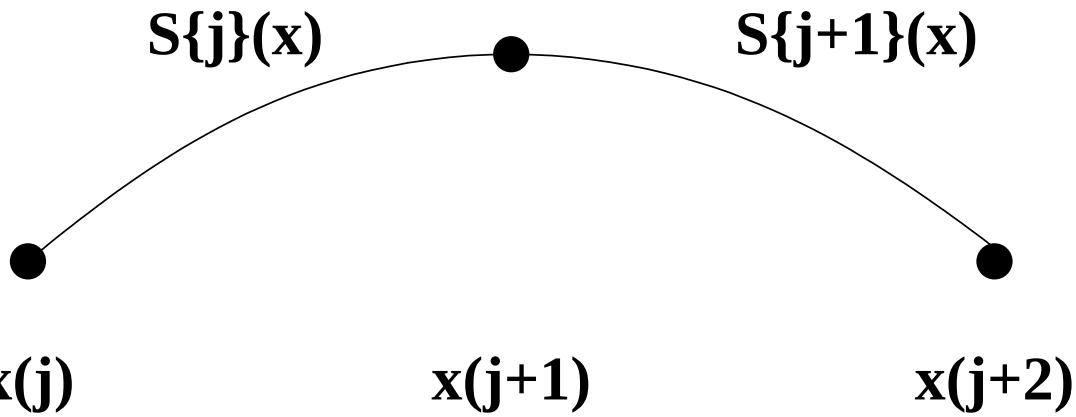
b. $S_j(x_j) = f(x_j),\ \forall j = 0, 1, \ldots, (n-1)$. ***"Left" Interpolation***

c. $S_j(x_{j+1}) = f(x_{j+1}),\ \forall j = 0, 1, \ldots, (n-1)$. ***"Right" Interpolation***

d. $S'_j(x_{j+1}) = S'_{j+1}(x_{j+1}),\ \forall j = 0, 1, \ldots, (n-2)$. ***Slope-match***

e. $S''_j(x_{j+1}) = S''_{j+1}(x_{j+1}),\ \forall j = 0, 1, \ldots, (n-2)$. ***Curvature-match***

A Spline Segment

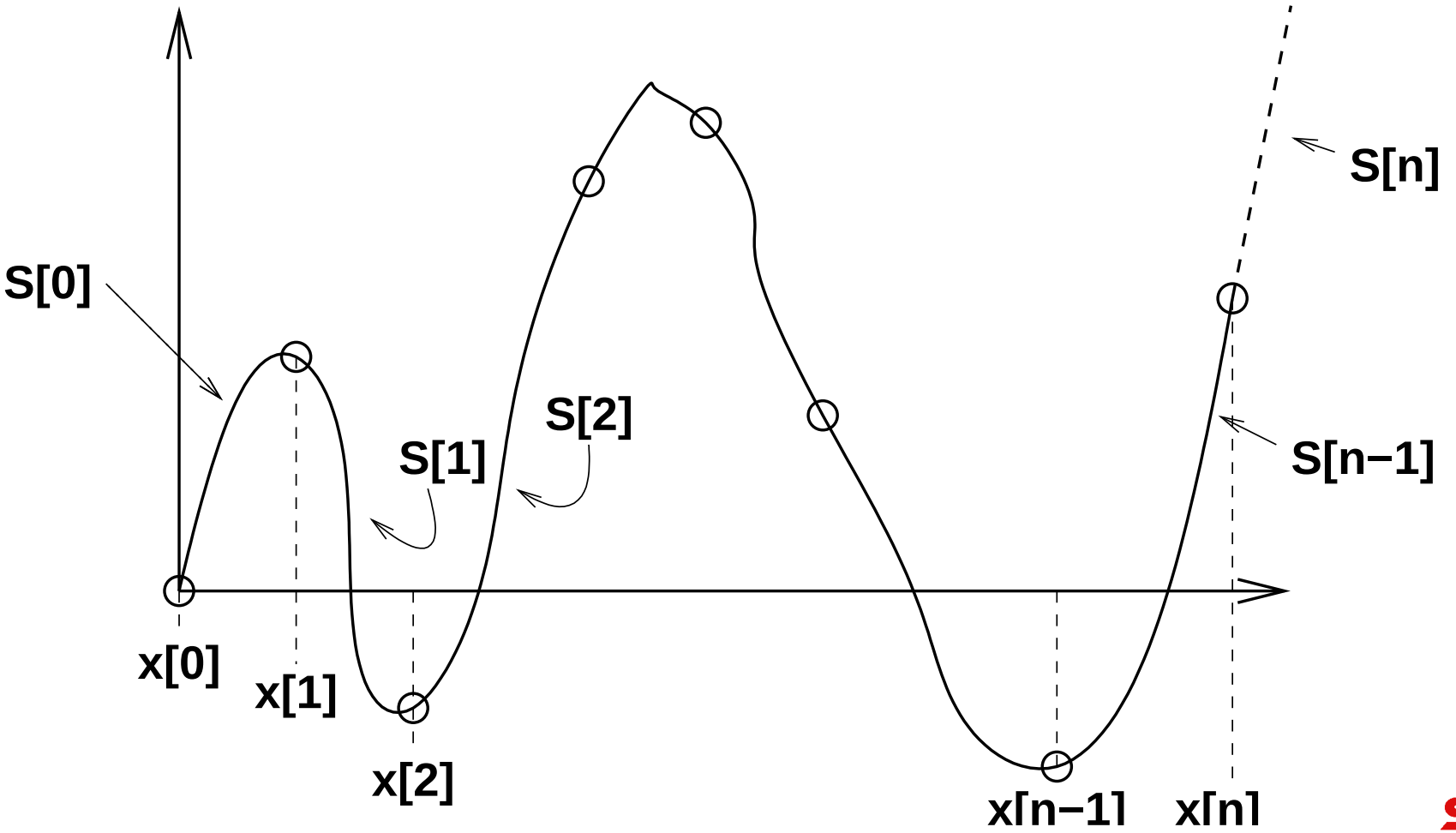


The spline segment $S_j(x)$ "lives" on the interval $[x_j, x_{j+1}]$.
The spline segment $S_{j+1}(x)$ "lives" on the interval $[x_{j+1}, x_{j+2}]$.

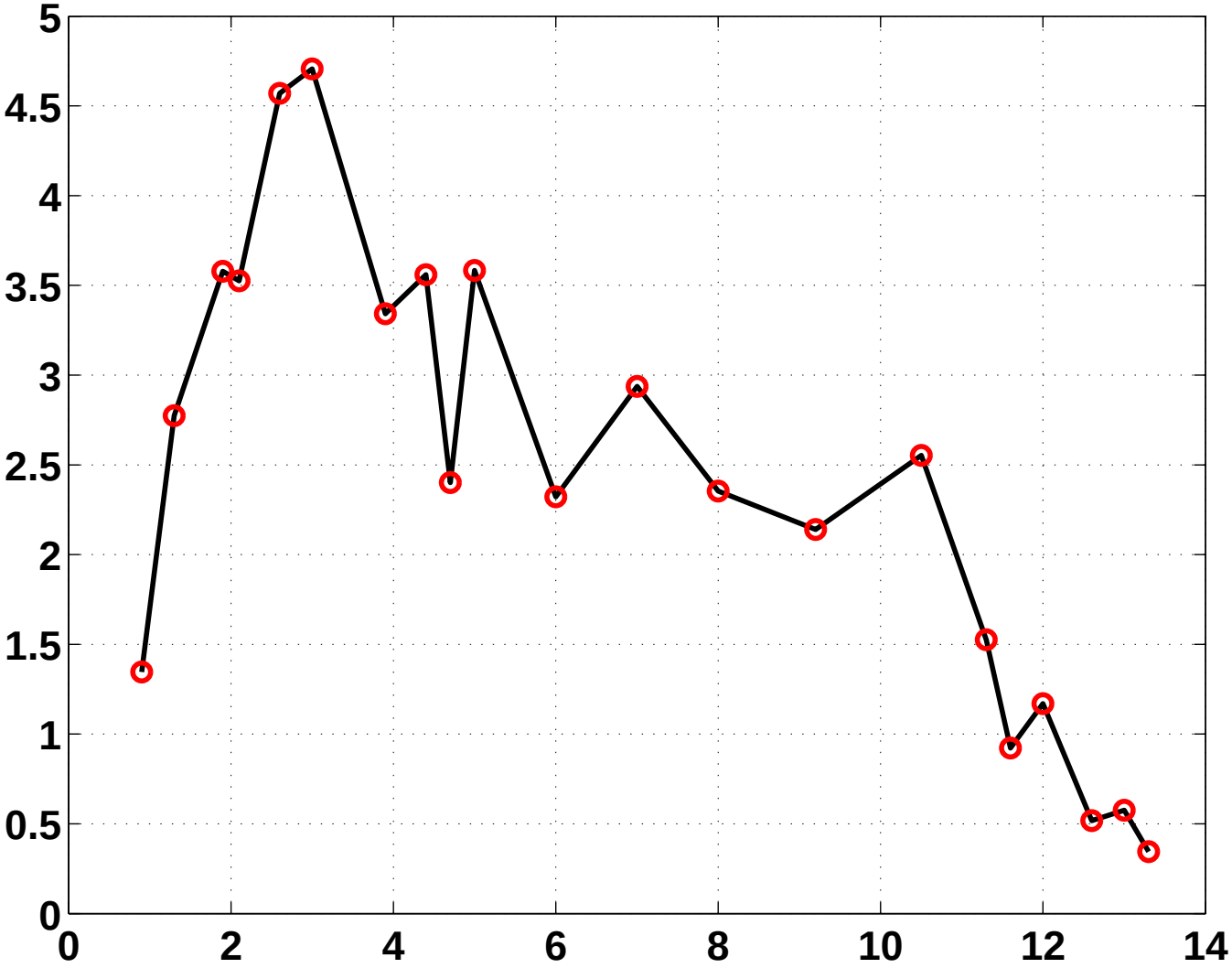
Their function values: $S_j(x_{j+1}) = S_{j+1}(x_{j+1}) = f(x_{j+1})$
derivatives: $S'_j(x_{j+1}) = S'_{j+1}(x_{j+1})$
and second derivatives: $S''_j(x_{j+1}) = S''_{j+1}(x_{j+1})$

... are required to match in the ***interior*** point x_{j+1}.

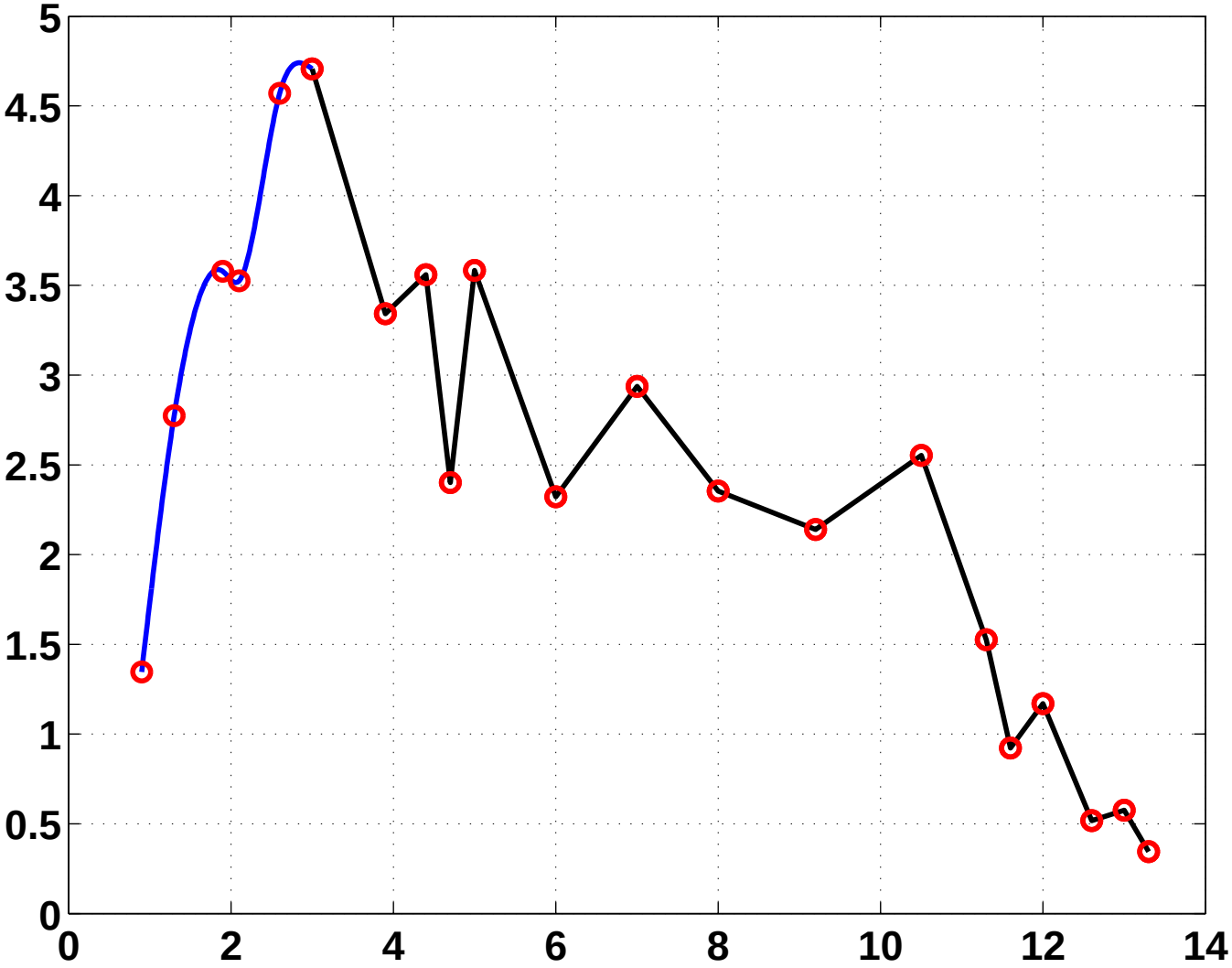
Example "Cartoon": Cubic Spline.



Example "Progressive" Cubic Spline Interpolation



Example "Progressive" Cubic Spline Interpolation



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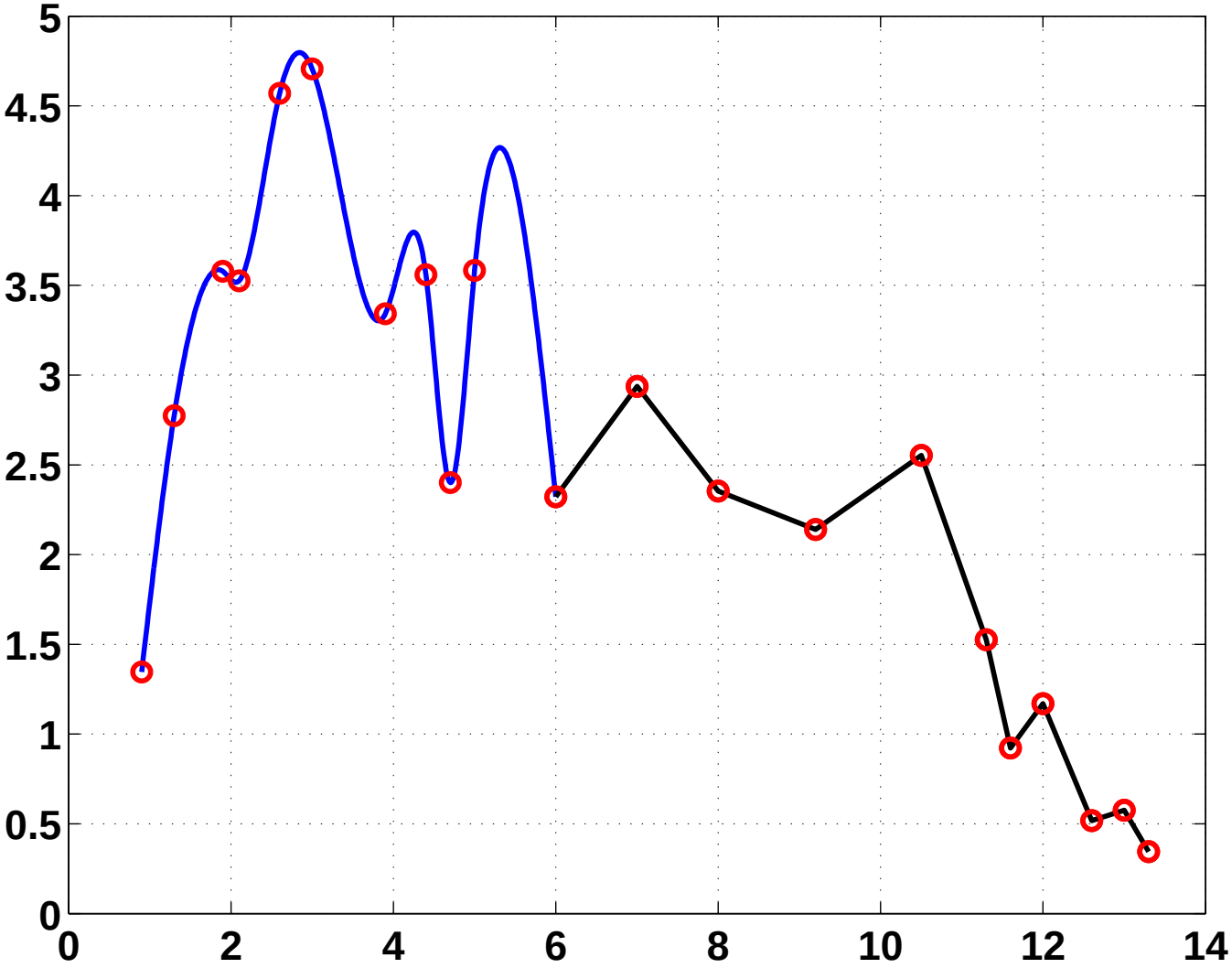

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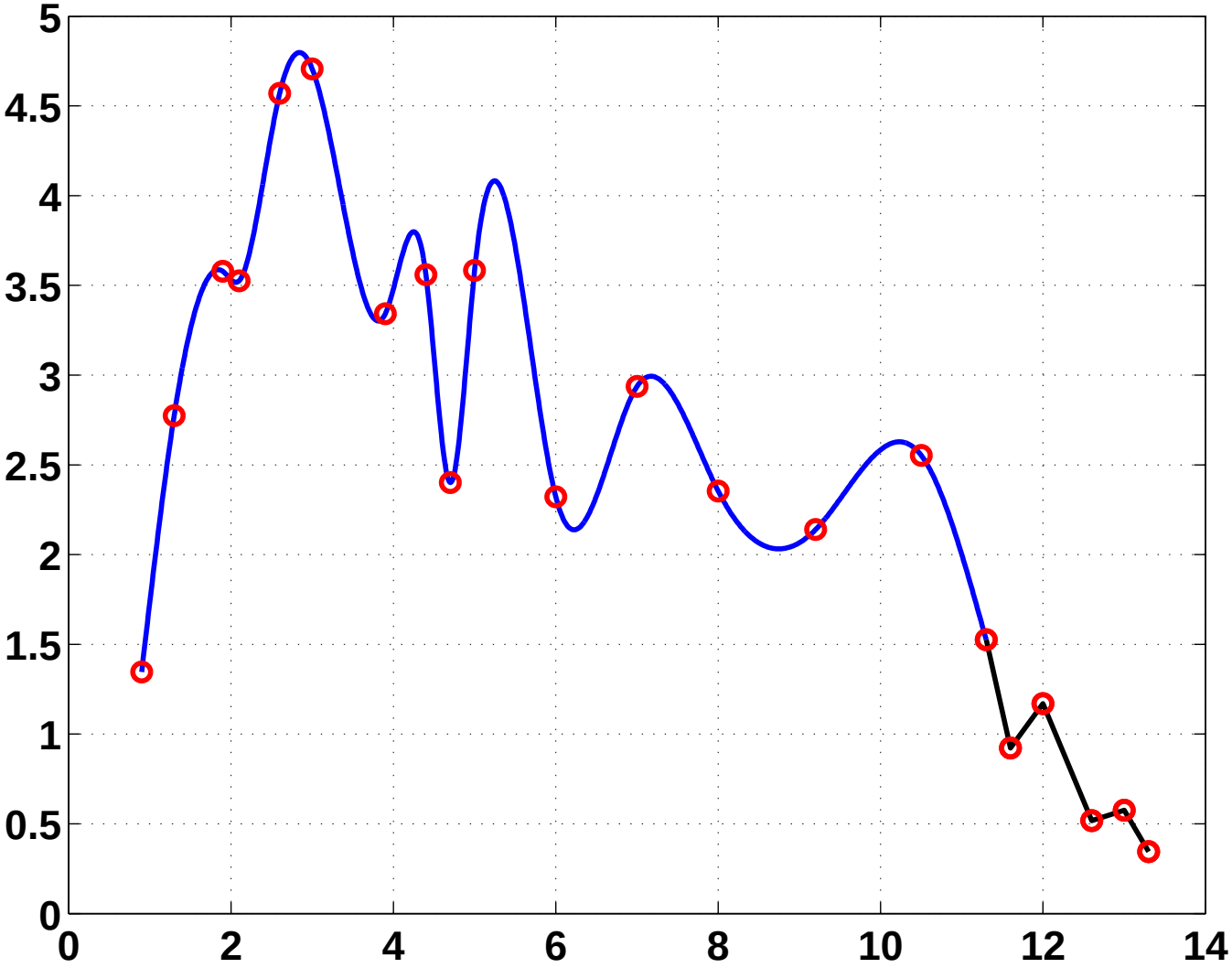

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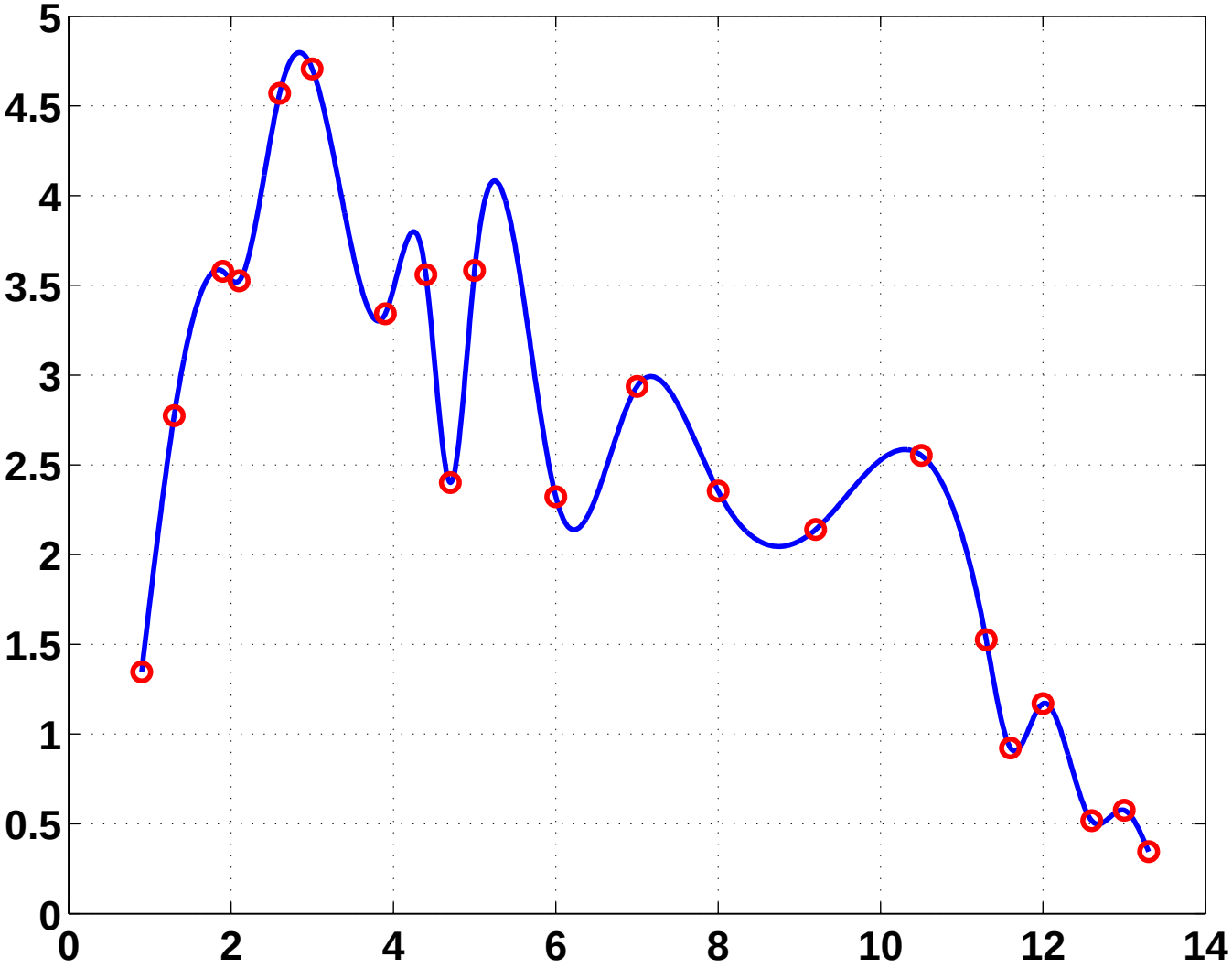

Building Cubic Splines, I. — Applying the Conditions

We start with

$$S_j(x) = a_j + b_j(x - x_j) + c_j(x - x_j)^2 + d_j(x - x_j)^3$$
$$\forall j \in \{0, 1, \ldots, n-1\}$$

and apply all the conditions to these polynomials...

For convenience we introduce the notation $h_j = x_{j+1} - x_j$.

b. $S_j(x_j) = a_j = f(x_j)$

c. $S_{j+1}(x_{j+1}) = \mathbf{a_{j+1} = a_j + b_j h_j + c_j h_j^2 + d_j h_j^3} = S_j(x_{j+1})$

d. Notice $S_j'(x_j) = b_j$, hence we get $\mathbf{b_{j+1} = b_j + 2c_j h_j + 3d_j h_j^2}$

e. Notice $S_j''(x_j) = 2c_j$, hence we get $\mathbf{c_{j+1} = c_j + 3d_j h_j}$.

— We got a whole lot of equations to solve!!! (How many???)

Cubic Splines, II. — Solving the Resulting Equations.

We solve [**e**] for $d_j = \dfrac{c_{j+1} - c_j}{3h_j}$, and plug into [**c**] and [**d**] to get

[**c'**] $a_{j+1} = a_j + b_j h_j + \dfrac{h_j^2}{3}(2c_j + c_{j+1})$,

[**d'**] $b_{j+1} = b_j + h_j(c_j + c_{j+1})$.

We solve for b_j in [**c'**] and get

[*] $b_j = \dfrac{1}{h_j}(a_{j+1} - a_j) - \dfrac{h_j}{3}(2c_j + c_{j+1})$.

Reduce the index by 1, to get

[*'] $b_{j-1} = \dfrac{1}{h_{j-1}}(a_j - a_{j-1}) - \dfrac{h_{j-1}}{3}(2c_{j-1} + c_j)$.

Plug [*] (lhs) and [*'] (rhs) into the index-reduced-by-1 version of [**d'**], *i.e.*

[**d''**] $b_j = b_{j-1} + h_{j-1}(c_{j-1} + c_j)$.

Cubic Splines, III. — A Linear System of Equations

After some "massaging" we end up with the linear system of equations for $j \in \{1, 2, \ldots, n-1\}$ (the interior nodes).

$$h_{j-1}c_{j-1}+2(h_{j-1}+h_j)c_j+h_jc_{j+1} = \frac{3}{h_j}(a_{j+1}-a_j)-\frac{3}{h_{j-1}}(a_j-a_{j-1}).$$

Notice: The only unknowns are $\{c_j\}_{j=0}^{n}$, since the values of $\{a_j\}_{j=0}^{n}$ and $\{h_j\}_{j=0}^{n-1}$ are given.

Once we compute $\{c_j\}_{j=0}^{n-1}$, we get

$$b_j = \frac{a_{j+1}-a_j}{h_j} - \frac{h_j(2c_j+c_{j+1})}{3}, \quad \text{and} \quad d_j = \frac{c_{j+1}-c_j}{3h_j}.$$

We are ***almost*** ready to solve for the coefficients $\{c_j\}_{j=0}^{n-1}$, but we only have $(n-1)$ equations for $(n+1)$ unknowns...

Cubic Splines, IV. — Completing the System

We can complete the system in many ways, some common ones are...

Natural boundary conditions:

$$[\mathbf{n1}] \quad 0 = S_0''(x_0) = 2c_0 \quad \Rightarrow \quad c_0 = 0$$

$$[\mathbf{n2}] \quad 0 = S_n''(x_n) = 2c_n \quad \Rightarrow \quad c_n = 0$$

Cubic Splines, IV. — Completing the System

We can complete the system in many ways, some common ones are...

Clamped boundary conditions: (Derivative known at endpoints).

$$\begin{aligned}
[\mathbf{c1}] \qquad S_0'(x_0) &= b_0 = f'(x_0) \\
[\mathbf{c2}] \quad S_{n-1}'(x_n) &= b_n = b_{n-1} + h_{n-1}(c_{n-1} + c_n) = f'(x_n)
\end{aligned}$$

[**c1**] and [**c2**] give the additional equations

$$\begin{aligned}
[\mathbf{c1'}] \qquad 2h_0c_0 + h_0c_1 &= \tfrac{3}{h_0}(a_1 - a_0) - 3f'(x_0) \\
[\mathbf{c2'}] \quad h_{n-1}c_{n-1} + 2h_{n-1}c_n &= 3f'(x_n) - \tfrac{3}{h_{n-1}}(a_n - a_{n-1}).
\end{aligned}$$

Cubic Splines, IV. — Completing the System

Given a function f defined on $[a, b]$ and a set of nodes $a = x_0 < x_1 < \cdots < x_n = b$, a ***cubic spline interpolant*** *S for f is a function that satisfies the following conditions:*

a. $S(x)$ *is a cubic polynomial, denoted* $S_j(x)$, *on the sub-interval* $[x_j, x_{j+1}]$ $\forall j = 0, 1, \ldots, n-1$.

b. $S_j(x_j) = f(x_j),\ \forall j = 0, 1, \ldots, (n-1)$. — ***"Left" Interpolation***

c. $S_j(x_{j+1}) = f(x_{j+1}),\ \forall j = 0, 1, \ldots, (n-1)$. — ***"Right" Interpolation***

d. $S'_j(x_{j+1}) = S'_{j+1}(x_{j+1}),\ \forall j = 0, 1, \ldots, (n-2)$. — ***Slope-match***

e. $S''_j(x_{j+1}) = S''_{j+1}(x_{j+1}),\ \forall j = 0, 1, \ldots, (n-2)$. — ***Curvature-match***

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f. *One of the following sets of boundary conditions is satisfied:*

1. $S''(x_0) = S''(x_n) = 0$, – ***free / natural boundary***
2. $S'(x_0) = f'(x_0)$ *and* $S'(x_n) = f'(x_n)$, – ***clamped boundary***

Natural Boundary Conditions: Linear System, $A\tilde{\mathbf{x}} = \tilde{\mathbf{y}}$

We end up with a linear system of equations, $A\tilde{\mathbf{x}} = \tilde{\mathbf{y}}$, where

$$A = \begin{bmatrix} \mathbf{1} & 0 & 0 & \cdots & \cdots & 0 \\ h_0 & 2(h_0+h_1) & h_1 & \ddots & & \vdots \\ 0 & h_1 & 2(h_1+h_2) & h_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & h_{n-2} & 2(h_{n-2}+h_{n-1}) & h_{n-1} \\ 0 & \cdots & \cdots & 0 & 0 & \mathbf{1} \end{bmatrix},$$

Boundary Terms: marked in red-bold.

Natural Boundary Conditions: Linear System, $A\tilde{\mathbf{x}} = \tilde{\mathbf{y}}$

We end up with a linear system of equations, $A\tilde{\mathbf{x}} = \tilde{\mathbf{y}}$, where

$$\tilde{\mathbf{y}} = \begin{bmatrix} \mathbf{0} \\ \frac{3(a_2 - a_1)}{h_1} - \frac{3(a_1 - a_0)}{h_0} \\ \vdots \\ \frac{3(a_n - a_{n-1})}{h_{n-1}} - \frac{3(a_{n-1} - a_{n-2})}{h_{n-2}} \\ \mathbf{0} \end{bmatrix}, \qquad \tilde{\mathbf{x}} = \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n-1} \\ c_n \end{bmatrix}$$

$\tilde{\mathbf{x}}$ are the unknowns (the quantity we are solving for!)

Boundary Terms: marked in red-bold.

Clamped Boundary Conditions: Linear System

We end up with a linear system of equations, $A\tilde{\mathbf{x}} = \tilde{\mathbf{y}}$, where

$$A = \begin{bmatrix} \mathbf{2h_0} & \mathbf{h_0} & 0 & \cdots & \cdots & 0 \\ h_0 & 2(h_0+h_1) & h_1 & \ddots & & \vdots \\ 0 & h_1 & 2(h_1+h_2) & h_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & h_{n-2} & 2(h_{n-2}+h_{n-1}) & h_{n-1} \\ 0 & \cdots & \cdots & 0 & \mathbf{h_{n-1}} & \mathbf{2h_{n-1}} \end{bmatrix}$$

Boundary Terms: marked in red-bold.

Cubic Spline Natural B.C.

MatLab code for a cubic spline with Natural Boundary conditions.

```
1  % Cubic Spline Interpolation - Natural Spline
2  % INPUT: xi is the grid (points on x-axis) and a ...
       are points on y-axis. inter
3  % is the point on the x-axis you want to know the ...
       value of on the y-axis.
4
5  function [P] = cubic_splinenat(xi,a)
6
7  if length(xi) ≠ length(a)
8   erro('vectors xi and a must be of same length');
9  end
10
11 % Plotting points we want to interpolate between:
12 grid on; hold on;
13 title('Cubic Spline Interpolation');
14 plot(xi,a,'or');
```

Cubic Spline Natural B.C.

```
16  n = length(xi);
17
18  % Vector h with subintervals:
19  h = zeros(n-1,1);
20  for j = 1:n-1
21   h(j) = xi(j+1) - xi(j);
22  end
23
24  % Coefficient matrix A:
25  A = zeros(n);
26
27  % Natural Spline boundary conditions:
28  A(1,1)= 1;
29  A(n,n) = 1;
```

Cubic Spline Natural B.C.

```
31  for i = 2:n-1
32   A(i,i-1) = h(i-1);
33   A(i,i) = 2*(h(i-1)+h(i));
34   A(i,i+1) = h(i);
35  end
36
37  % Vector b:
38  b = zeros(n,1);
39
40  for i = 2:n-1
41   b(i) = (3/h(i))*(a(i+1)-a(i)) - ...
          (3/h(i-1))*(a(i)-a(i-1));
42  end
43
44  % Coefficient vector cj:
45  cj = A\b;
```

Cubic Spline Natural B.C.

```
47  % Coefficient vector bj:
48  bj = zeros(n-1,1);
49  for i = 1:n-1
50   bj(i) = (1/h(i))*(a(i+1)-a(i)) - ...
         (1/3*h(i))*(2*cj(i)+cj(i+1));
51  end
52
53  % Coefficient vector dj:
54  dj = zeros(n-1,1);
55  for i = 1:n-1
56   dj(i) = (1/(3*h(i))) * (cj(i+1)-cj(i));
57  end
58
59  % Making a matrix P with all polynomials (and ...
       intervals)
60  P = zeros(n-1,4);
61  X = zeros(n-1,2);
```

Cubic Spline Natural B.C.

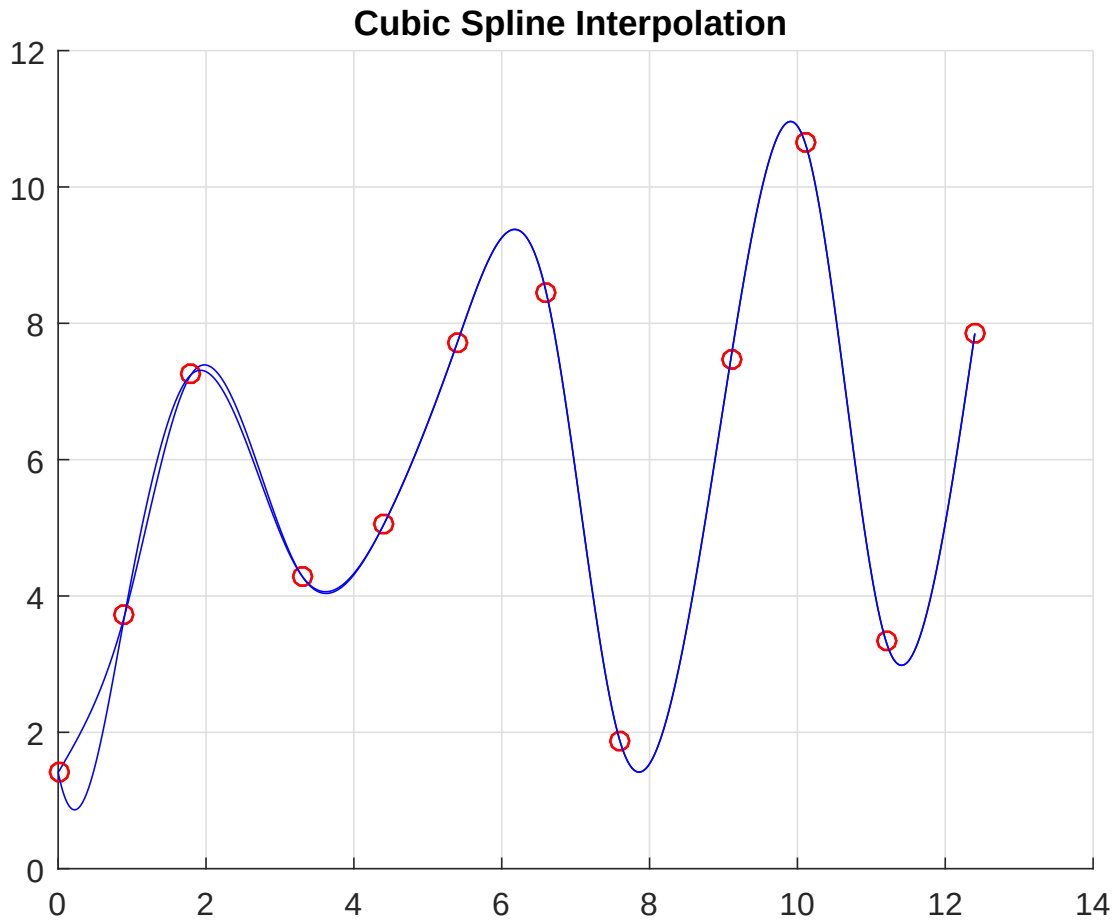
```
62  for i = 1:n-1
63   P(i,1) = dj(i);
64   P(i,2) = cj(i);
65   P(i,3) = bj(i);
66   P(i,4) = a(i);
67   X(i,1) = xi(i);
68   X(i,2) = xi(i+1);
69  end
70   P = [P,X];
```

Cubic Spline Natural B.C.

```
72  % Plotting results:
73  resolution = 100;
74  for i = 1:n-1
75   f = @(x) a(i) + bj(i).*(x-xi(i)) + ...
         cj(i).*(x-xi(i)).^2 + dj(i).*(x-xi(i)).^3;
76   xf = linspace(xi(i),xi(i+1),resolution);
77   plot(xf,f(xf),'b-');
78  end
79  %print -depsc spline_nat.eps
80  end
```

Cubic Spline Natural B.C.

Cubic Spline with Natural boundary conditions



Clamped Boundary Conditions: Linear System

We end up with a linear system of equations, $A\tilde{\mathbf{x}} = \tilde{\mathbf{y}}$, where

$$\tilde{\mathbf{y}} = \begin{bmatrix} \mathbf{\frac{3(a_1-a_0)}{h_0} - 3f'(x_0)} \\ \frac{3(a_2-a_1)}{h_1} - \frac{3(a_1-a_0)}{h_0} \\ \vdots \\ \frac{3(a_n-a_{n-1})}{h_{n-1}} - \frac{3(a_{n-1}-a_{n-2})}{h_{n-2}} \\ \mathbf{3f'(x_n) - \frac{3(a_n-a_{n-1})}{h_{n-1}}} \end{bmatrix}, \quad \tilde{\mathbf{x}} = \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n-1} \\ c_n \end{bmatrix}$$

Boundary Terms: marked in red-bold.

Cubic Spline clamped B.C.

MatLab code changes for a cubic spline with Clamped Boundary conditions.

```
1 % Cubic Spline Interpolation - Clamped
2 % INPUT: xi is the grid (points on x-axis) and a ...
      are points on y-axis. inter
3 % is the point on the x-axis you want to know the ...
      value of on the y-axis.
4
5 function [P] = cubic_splineclp(xi,a,fp)
```

Cubic Spline clamped B.C.

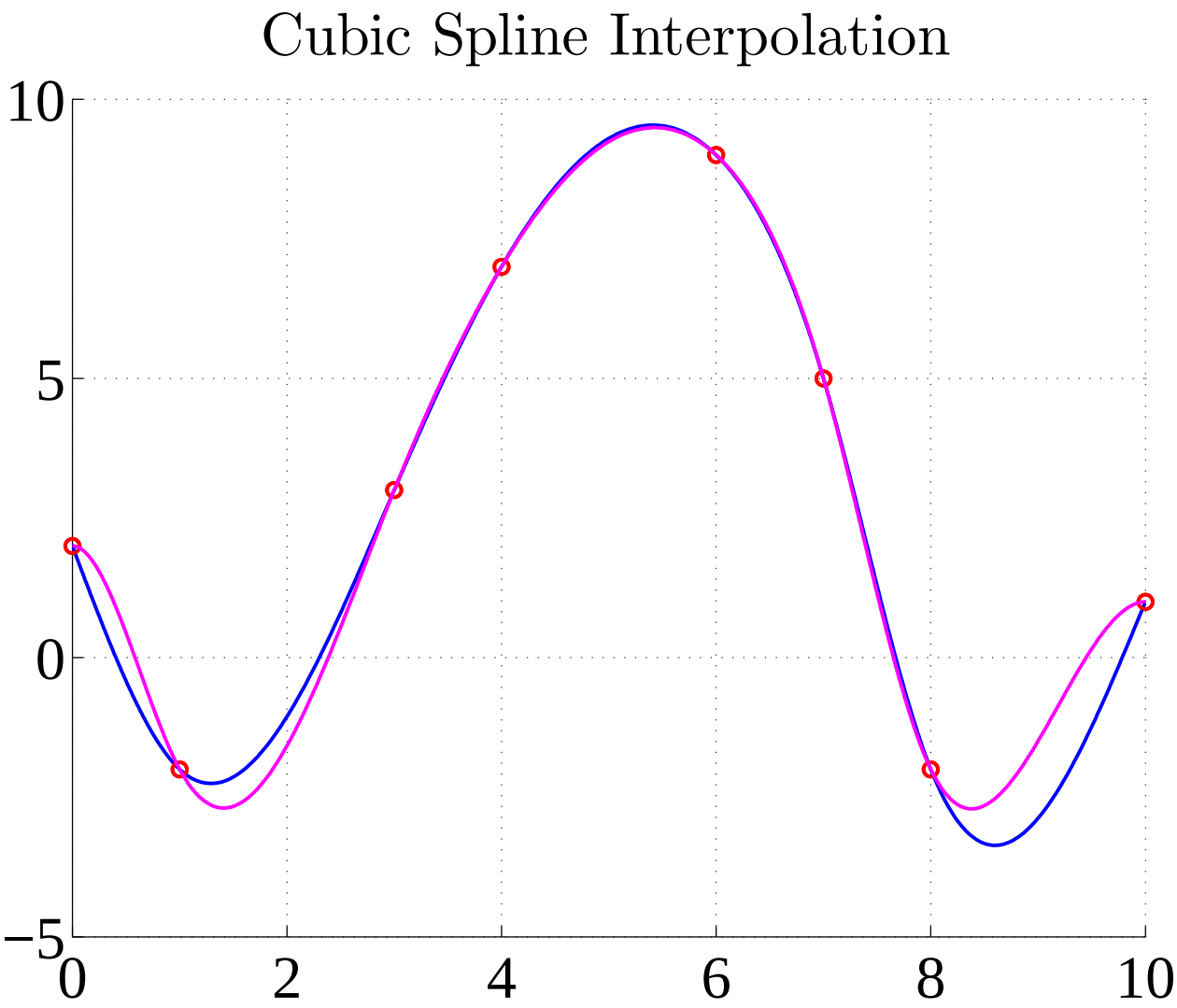
More **MatLab code changes** for a cubic spline with Clamped Boundary conditions.

```
27 % Clamped boundary conditions:
28 A(1,1) = 2*h(1);
29 A(1,2) = h(1);
30 A(n,n) = 2*h(n-1);
31 A(n,n-1) = h(n-1);
```

```
39 % Vector b:
40 b = zeros(n,1);
41 b(1) = (3/h(1))*(a(2)-a(1)) - 3*fp(1);
42 b(n) = 3*fp(2) - (3/h(n-1))*(a(n)-a(n-1));
```

Cubic Spline clamped B.C.

Cubic Spline with clamped boundary conditions



Cubic Splines, The Error Bound

No numerical story is complete without an error bound...
If $f \in C^4[a,b]$, *let*

$$M = \max_{a \le x \le b} |f^4(x)|.$$

If S *is the unique* ***clamped cubic spline interpolant*** *to* f *with respect to the nodes* $a = x_0 < x_1 < \cdots < x_n = b$, *then with*

$$h = \max_{0 \le j \le n-1} (x_{j+1} - x_j) = \max_{0 \le j \le n-1} h_j$$

$$\max_{a \le x \le b} |f(x) - S(x)| \le \frac{5Mh^4}{384}$$

Banded Matrices [Reference]

We notice that the linear systems for both natural and clamped boundary conditions give rise to ***tri-diagonal linear systems***.

Further, these systems are ***strictly diagonally dominant*** — the entries on the diagonal outweigh the sum of the off-diagonal elements (in absolute terms) —, so pivoting (re-arrangement to avoid division by a small number) is not needed when solving for $\tilde{\mathbf{x}}$ using Gaussian Elimination...

This means that these systems can be solved very quickly (we will revisit this topic later on, but for now the algorithm is on the next couple of slides), see also *"Computational Linear Algebra / Numerical Matrix Analysis."* Math 543

Algorithm: Solving $Tx = b$ in $\mathcal{O}(n)$ Time, I. [Reference]

The Thomas Algorithm

Given the $N \times N$ tridiagonal matrix T and the $N \times 1$ vector y:

Step 1: The first row:

$$\begin{aligned} l_{1,1} &= T_{1,1} \\ u_{1,2} &= T_{1,2}/l_{1,1} \\ z_1 &= y_1/l_{1,1} \end{aligned}$$

Step 2: FOR $i = 2 : (n-1)$

$$\begin{aligned} l_{i,i-1} &= T_{i,i-1} \\ l_{i,i} &= T_{i,i} - l_{i,i-1}u_{i-1,i} \\ u_{i,i+1} &= T_{i,i+1}/l_{i,i} \\ z_i &= (y_i - l_{i,i-1}z_{i-1})/l_{i,i} \end{aligned}$$

END

Step 3: The last row:

$$\begin{aligned} l_{n,n-1} &= T_{n,n-1} \\ l_{n,n} &= T_{n,n} - l_{n,n-1}u_{n-1,n} \\ z_n &= (y_n - l_{n,n-1}z_{n-1})/l_{n,n} \end{aligned}$$

Step 4: $x_n = z_n$

Step 5: FOR $i = (n-1) : -1 : 1$

$$x_i = z_i - u_{i,i+1}x_{i+1}$$

END

The Thomas Algorithm: Solving $Tx = b$ in $\mathcal{O}(n)$ Time, II. [Reference]

The algorithm computes both the LU-factorization of T, as well as the solution $\tilde{\mathbf{x}} = T^{-1}\tilde{\mathbf{y}}$. Steps 1–3 computes $\tilde{\mathbf{z}} = L^{-1}\tilde{\mathbf{y}}$, and steps 4–5 computes $\tilde{\mathbf{x}} = U^{-1}\tilde{\mathbf{z}}$.

MatLab Tridiagonal Solver

It is more efficient to store our matrix A as only 3 vectors for large A

Below is a more efficient way to store and solve a **Tridiagonal matrix**, assuming it is tridiagonal

```
1  function y = tridiag( a, b, c, f )
2
3  %  Solve the  n x n  tridiagonal system Ay = f ...
       for y
4  %  with diagonal a, subdiagonal b, and ...
       superdiagonal c
5  %  f must be a vector (row or column) of length n
6  %  a, b, c must be vectors of length n
7  % (note that b(1) and c(n) are not used)
```

MatLab Tridiagonal Solver

Below is the main code for solving a **Tridiagonal matrix**, assuming it is tridiagonal

```
20  n = length(f);
21  v = zeros(n,1);
22  y = v;
23  w = a(1);
24  y(1) = f(1)/w;
25  for i=2:n
26      v(i-1) = c(i-1)/w;
27      w = a(i) - b(i)*v(i-1);
28      y(i) = ( f(i) - b(i)*y(i-1) )/w;
29  end
30  for j=n-1:-1:1
31     y(j) = y(j) - v(j)*y(j+1);
32  end
```